

# CLASSIFICATION THEOREMS FOR CIRCLE ACTIONS ON KIRCHBERG ALGEBRAS, I

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**ABSTRACT.** We investigate basic properties of circle actions with the Rokhlin property. Using equivariant semiprojectivity and partial automorphisms, we show that every circle action with the Rokhlin property is a dual action. We also prove that Kirchberg algebras are preserved under formation of crossed products by such actions. It is shown that the Rokhlin property implies severe  $K$ -theoretical constraints on the algebra it acts on. We explain how circle actions with the Rokhlin property have an  $\text{Ext}(K_*(A^{\mathbb{T}}), K_{*+1}(A^{\mathbb{T}}))$ -class naturally associated to them. If  $A$  is a Kirchberg algebra satisfying the UCT, this Ext-class is shown to be a complete invariant for such actions up to conjugacy. More generally, we show that circle actions with the Rokhlin property on Kirchberg algebras are uniquely determined by the  $KK$ -class of their predual automorphisms.

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## 1. INTRODUCTION

Elliott's classification program has as its ultimate goal the classification of all (separable) nuclear, simple  $C^*$ -algebras in terms of what is now known as the Elliott invariant, which is essentially  $K$ -theoretical in nature. This classification program was initiated by George Elliott with the classification of AF-algebras, and extended by himself with the classification of certain simple  $C^*$ -algebras of real rank zero. These early successes were quickly followed by many other classification results for

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nuclear  $C^*$ -algebras.

Crossed product  $C^*$ -algebras have provided some of the most relevant examples in the theory of  $C^*$ -algebras, and the classification of crossed products, or at least the study of their structure, is a very active field of research. Moreover, the classification of group actions on  $C^*$ -algebras, which is in general far more difficult than the classification of their crossed products, has always received a great deal of attention. Early results in this direction include work of Herman and Ocneanu ([HO84]), work of Fack and Maréchal ([FM81] and [FM]), and the work of Handelman and Rossmann ([HR85]). More recent results have been obtained by Izumi in [Izu04a] and [Izu04b], where he systematically studied the classification problem for finite group actions with the Rokhlin property on  $C^*$ -algebras. When trying to classify actions on  $C^*$ -algebras, one usually has to restrict oneself to a specific classifiable class of  $C^*$ -algebras, and also focus on a specific class of actions on them. The main feature that distinguishes the class of actions on which Izumi focused, is the fact that they are not specified by the way they are constructed. Indeed, the previous known results only considered a rather limited class of finite (and sometimes compact) group actions, namely the class of locally representable actions on AF-algebras, which in particular are direct limit actions. The class of actions considered by Izumi is the class of finite group actions with the Rokhlin property. When the group is (finite and) abelian, the dual action of a locally representable action has the Rokhlin property. This explains why locally representable actions can be classified in terms of their crossed products and dual actions. In fact, Izumi obtains most of the classification theorems of the other cited authors as consequences of his results.

It is in this sense that the Rokhlin property is the main assumption in the majority of the classification results for finite group actions so far available. This should come as no surprise, in the light of Connes' classification of outer actions of amenable discrete groups on von Neumann algebras in [Con75], where the main technical tool is the fact that all such actions have the Rokhlin property.

In [HW07], Hirshberg and Winter extended Izumi's definition of a finite group action with the Rokhlin property to the second countable, compact group case, and proved that absorption of a strongly self absorbing  $C^*$ -algebra and approximate divisibility pass to crossed products by such actions.

It is natural to explore the classification of Rokhlin actions of compact groups on classifiable classes of  $C^*$ -algebras, generalizing work of Izumi in the finite group case. We began exploring this problem in [Gar14a], where we considered circle actions on (not necessarily simple) nuclear separable unital  $\mathcal{O}_2$ -absorbing  $C^*$ -algebras. This work focuses on the classification problem for circle actions with the Rokhlin property on purely infinite simple  $C^*$ -algebras. A  $C^*$ -algebra is said to be a *Kirchberg algebra* if it is purely infinite, simple, separable and nuclear. (This terminology was first introduced by Rørdam in [Rør02] to recognize the significant contribution of Eberhard Kirchberg to the study and classification of these algebras, which he originally called *pisun*.)

We point out that, in view of the results in [OP12], another interesting problem is the study of the structure of the crossed product by a compact group action with the Rokhlin property. This problem is studied in [Gar14c].

The classification results in [Kir94] and [Phi00] can be thought of as a starting point for this work, so we briefly recall them. If  $A$  and  $B$  are two unital Kirchberg

algebras, then  $A$  and  $B$  are isomorphic if and only if there is an invertible element in  $KK(A, B)$  that respects the classes of the units of  $A$  and  $B$ . If  $A$  and  $B$  moreover satisfy the Universal Coefficient Theorem (UCT), then for every  $\mathbb{Z}_2$ -graded isomorphism  $\varphi_*: K_*(A) \rightarrow K_*(B)$  satisfying  $\varphi_0([1_A]) = [1_B]$ , there is an isomorphism  $\psi: A \rightarrow B$  such that  $K_*(\psi) = \varphi_*$ . Furthermore, every pair of countable abelian groups arises as the  $K$ -groups of a unital Kirchberg algebra that satisfies the UCT, and the class of the unit of the algebra in  $K_0$  can be arbitrary.

This paper is organized as follows. In Section 2, we introduce the notation that will be used throughout the paper. We also present there the necessary background on partial automorphisms and their crossed products, and on equivariant  $K$ -theory. In Section 3, we develop a systematic study of circle actions with the Rokhlin property on unital  $C^*$ -algebras. Propositions 3.6 and 3.7 establish the duality between the Rokhlin property and approximate representability, extending the results of [Izu04a] in the finite group case. Simplicity of the crossed product is studied in Proposition 3.10. Moreover, in Theorem 3.11 we use equivariant semiprojectivity and results of Exel from [Exe94] on partial automorphisms, to show that every circle action with the Rokhlin property is conjugate to the dual action of an automorphism of its fixed point algebra. In Section 4, we develop an averaging technique using the Rokhlin property that will allow us to define, given a compact subset  $F$  of  $A$ , a linear map  $A \rightarrow A^\alpha$  which is an approximate  $*$ -homomorphism on  $F$ , in a suitable sense. See Theorem 4.3. In Section 5, we apply our averaging technique to study the  $K$ -theory of the crossed product; see Theorems 5.2 and 5.5. In Section 6, we specialize to purely infinite simple  $C^*$ -algebras, and show in Theorem 6.4 that the crossed product and fixed point algebra of a circle action with the Rokhlin property on a Kirchberg algebra, are again Kirchberg algebras. We then proceed to prove classification results for circle actions with the Rokhlin property on such algebras. See Theorem 6.7 for the general Kirchberg algebra case, and Theorem 6.9 for the Kirchberg UCT case.

Some of the results in this paper, specifically those in Section 4, are given in greater generality than needed here because the extra flexibility is needed in [Gar14b], where we show that in the presence of what we call the *continuous* Rokhlin property, there is an asymptotic homomorphism  $A \rightarrow A^\alpha$  which is a left inverse of the canonical inclusion  $A^\alpha \rightarrow A$ . In Section 7, we motivate some connections with the second part of this work, and we also give some indications of the difficulties of adapting the techniques used in this paper to the classification of actions of other compact Lie groups.

While working on this project, we learned that Rasmus Bentmann and Ralf Meyer have developed techniques that allow them to classify objects in triangulated categories with projective resolutions of length two. See [BM14]. Their study applies to circle actions on  $C^*$ -algebras, the invariant being equivariant  $K$ -theory, and isomorphism of actions being  $KK^\mathbb{T}$ -equivalence. Their results predict the same outcome that we have obtained, at least up to  $KK^\mathbb{T}$ -equivalence. Some work has to be done to deduce conjugacy from  $KK^\mathbb{T}$ -equivalence for circle actions with the Rokhlin property on Kirchberg algebras. The fact that the corresponding non-equivariant statement is true, as was shown in Corollary 4.2.2 in [Phi00], and also [Kir94], strongly suggests that this ought to be true in the equivariant case as well.

In Subsection 6.1, we include some comments on how the work of Bentmann-Meyer could be used to obtain  $KK^{\mathbb{T}}$ -equivalence in the cases we consider.

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## 2. NOTATION AND PRELIMINARIES

We adopt the convention that  $\{0\}$  is not a unital  $C^*$ -algebra, this is, we require that  $1 \neq 0$  in a unital  $C^*$ -algebra. For a  $C^*$ -algebra  $A$ , we denote by  $\text{Aut}(A)$  the automorphism group of  $A$ . If  $A$  is moreover unital, then  $\mathcal{U}(A)$  denotes the unitary group of  $A$ , and two automorphisms  $\varphi$  and  $\psi$  of  $A$  are said to be approximately unitarily equivalent if  $\varphi \circ \psi^{-1}$  is approximately inner.

For a locally compact group  $G$ , an action of  $G$  on  $A$  is always assumed to be a *continuous* group homomorphism from  $G$  into  $\text{Aut}(A)$ , unless otherwise stated. If  $\alpha: G \rightarrow \text{Aut}(A)$  is an action of  $G$  on  $A$ , then we will denote by  $A^\alpha$ , and sometimes by  $A^G$ , the fixed point subalgebra of  $A$  under  $\alpha$ .

If  $G$  is a locally compact group, we denote by  $\text{Lt}: G \rightarrow \text{Aut}(C_0(G))$  the action given by left translation on  $G$ , this is,

$$\text{Lt}_g(f)(h) = f(g^{-1}h)$$

for all  $g, h$  in  $G$  and all  $f$  in  $C_0(G)$ . With a slight abuse of notation, we also denote by  $\text{Lt}$  the action of  $G$  on *itself* by left translation. The circle group will be denoted by  $\mathbb{T}$ .

We take  $\mathbb{N} = \{1, 2, \dots\}$ . For a  $C^*$ -algebra  $A$ , we set

$$\begin{aligned} \ell^\infty(\mathbb{N}, A) &= \left\{ (a_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}} : \sup_{n \in \mathbb{N}} \|a_n\| < \infty \right\}; \\ c_0(\mathbb{N}, A) &= \left\{ (a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n \rightarrow \infty} \|a_n\| = 0 \right\}; \\ A_\infty &= \ell^\infty(\mathbb{N}, A) / c_0(\mathbb{N}, A). \end{aligned}$$

We identify  $A$  with the constant sequences in  $\ell^\infty(\mathbb{N}, A)$  and with their image in  $A_\infty$ . We write  $A_\infty \cap A'$  for the central sequence algebra of  $A$ , that is, the relative commutant of  $A$  in  $A_\infty$ . For a sequence  $(a_n)_{n \in \mathbb{N}}$  in  $A$ , we denote by  $\overline{(a_n)}_{n \in \mathbb{N}}$  its image in  $A_\infty$ . We thus have

$$A_\infty = \left\{ \overline{(a_n)}_{n \in \mathbb{N}} \in A_\infty : \lim_{n \rightarrow \infty} \|a_n a - a a_n\| = 0 \text{ for all } a \in A \right\}.$$

If  $\alpha: G \rightarrow \text{Aut}(A)$  is an action of  $G$  on  $A$ , then there are actions of  $G$  on  $A_\infty$  and on  $A_\infty \cap A'$ , both denoted by  $\alpha_\infty$ . Note that unless the group  $G$  is discrete, these actions will in general not be continuous.

Given  $n \in \{2, \dots, \infty\}$ , we denote by  $\mathcal{O}_n$  the Cuntz algebra with canonical generators  $\{s_j\}_{j=1}^n$  satisfying the usual relations (see for example Section 4.2 in [Rør02]). Given  $n$  in  $\mathbb{N}$ , we denote by  $M_n$  the  $n$  by  $n$  matrices with complex coefficients, and by  $M_\infty$  the union

$$M_\infty = \bigcup_{n \in \mathbb{N}} M_n,$$

where we identify a matrix  $a$  in  $M_n$  with the matrix  $\text{diag}(a, 0)$  in  $M_{n+1}$  for all  $n$  in  $\mathbb{N}$ .

**2.1. Partial automorphisms and their crossed products.** One of our most crucial results states that every circle action with the Rokhlin property arises as a dual action. See Theorem 3.11. Our proof uses, among other things, machinery developed by Ruy Exel in [Exe94]. We proceed to briefly present it here.

**Definition 2.1.** (Definition 3.1 in [Exe94]) Let  $A$  be a  $C^*$ -algebra. A *partial automorphism* of  $A$  is a triple  $\Theta = (\theta, I, J)$ , where  $I$  and  $J$  are ideals in  $A$  and  $\theta: I \rightarrow J$  is a  $C^*$ -algebra isomorphism.

Given a  $C^*$ -algebra  $A$ , an action  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  and an integer  $n$ , recall that the  $n$ th spectral subspace of  $A$  is

$$A_n = \{a \in A: \alpha_\zeta(a) = \zeta^n a \text{ for all } \zeta \in \mathbb{T}\}.$$

Exel has shown that under mild conditions, any circle action on a  $C^*$ -algebra is conjugate to the dual action on a partial crossed product. The precise statement is reproduced below in the form of Theorem 2.4. We define these “mild conditions” first.

**Definition 2.2.** (Definitions 4.1 and 4.4 of [Exe94]) Let  $A$  be a  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action of the circle on  $A$ .

- (1) The action  $\alpha$  is said to be *semisaturated* if  $A$  is generated, as a  $C^*$ -algebra, by the union of the fixed point algebra  $A^\alpha = A_0$  and the first spectral subspace  $A_1$ .
- (2) The action  $\alpha$  is said to be *regular* if there exist an isomorphism  $\theta: A_1^* A_1 \rightarrow A_1 A_1^*$  and a linear isometry  $\lambda: A_1^* \rightarrow A_1 A_1^*$  such that for  $x, y \in A_1$ , for  $a \in A_1^* A_1$  and for  $b \in A_1 A_1^*$ , the following conditions hold:
  - (a)  $\lambda(x^* b) = \lambda(x^*) b$ ;
  - (b)  $\lambda(a x^*) = \theta(a) \lambda(x^*)$ ;
  - (c)  $\lambda(x^*)^* \lambda(y^*) = x y^*$ ; and
  - (d)  $\lambda(x^*) \lambda(y^*)^* = \theta(x^* y)$ .

Let  $A$  be a  $C^*$ -algebra and let  $\Theta = (\theta, I, J)$  be a partial automorphism of  $A$ ; see Definition 2.1. For  $n \in \mathbb{Z}$ , denote by  $D_n$  the domain of  $\theta^{-n}$ , or, equivalently, the image of  $\theta^n$ . Denote by  $L$  the subspace of  $\ell^1(\mathbb{Z}, A)$  consisting of all summable sequences  $(a(n))_{n \in \mathbb{Z}}$  in  $A$  such that  $a(n) \in D_n$  for each  $n \in \mathbb{Z}$ . In [Exe94], Exel defined a normed  $*$ -algebra structure on  $L$  as follows. (See comments below Proposition 3.3 in [Exe94].) For  $a$  and  $b$  in  $L$  and for  $n$  in  $\mathbb{Z}$ , set

$$(a * b)(n) = \sum_{k \in \mathbb{Z}} \theta^k(\theta^{-k}(a(k))b(n-k))$$

$$a^*(n) = \theta^n(a(-n)^*)$$

$$\|a\| = \sum_{n \in \mathbb{Z}} \|a(n)\|.$$

With the product, involution and norm defined above,  $L$  becomes a Banach  $*$ -algebra. See Theorem 3.6 in [Exe94].

In analogy to the usual definition of crossed product by a discrete group, the *partial crossed product* of  $A$  by  $\Theta$ , denoted  $A \rtimes_{\Theta} \mathbb{Z}$ , is defined to be the enveloping  $C^*$ -algebra of  $L$ . See Definition 3.7 in [Exe94].

**Remark 2.3.** Let  $A$  be a  $C^*$ -algebra and let  $\theta$  be an automorphism of  $A$ . Set  $\Theta = (\theta, A, A)$ . Then the partial crossed product  $A \rtimes_{\Theta} \mathbb{Z}$  is isomorphic to the usual crossed product  $A \rtimes_{\theta} \mathbb{Z}$  by the automorphism  $\theta$ . Indeed, if  $\theta$  is assumed to be an automorphism of  $A$ , then  $L$  is all of  $\ell^1(\mathbb{Z}, A)$ , and the operations defined on  $L$  defined above become the usual convolution, involution and norm on  $\ell^1(\mathbb{Z}, A)$ , whose enveloping  $C^*$ -algebra is by definition the crossed product of  $A$  by  $\theta$ .

The following is the main result in Section 4 of [Exe94].

**Theorem 2.4.** (Theorem 4.21 in [Exe94]) Let  $A$  be a  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action of the circle on  $A$ . Assume that  $\alpha$  is semi-saturated and regular. If  $\Theta = (\theta, A_1^* A_1, A_1 A_1^*)$  is the partial automorphism of the fixed point algebra  $A^{\alpha} = A_0$  as in the definition of regularity for circle actions above, then there exists an isomorphism

$$\varphi: A^{\alpha} \rtimes_{\Theta} \mathbb{Z} \rightarrow A,$$

that intertwines the dual action of  $\theta$  and  $\alpha$  and is the identity on  $A^{\alpha}$ .

**2.2. Equivariant  $K$ -theory and crossed products.** We devote this subsection to recalling the definition and some basic facts about equivariant  $K$ -theory. A thorough developement can be found in [Phi87], whose notation we will follow. We denote the suspension of a  $C^*$ -algebra  $A$  by  $SA$ .

**Definition 2.5.** Let  $G$  be a compact group and let  $(G, A, \alpha)$  be a unital  $G$ -algebra. Let  $\mathcal{P}_G(A)$  be the set of all  $G$ -invariant projections in all of the algebras  $\mathcal{B}(V) \otimes A$ , for all unitary finite dimensional representations  $\lambda: G \rightarrow \mathcal{U}(V)$ , the  $G$ -action on  $\mathcal{B}(V) \otimes A$  being the diagonal action, this is, the one determined by  $g \mapsto \text{Ad}(\lambda(g)) \otimes \alpha_g$  for  $g \in G$ . Note that there is no ambiguity about the tensor product norm on  $\mathcal{B}(V) \otimes A$  since  $V$  is finite dimensional.

Two  $G$ -invariant projections  $p$  and  $q \in \mathcal{P}_G(A)$  are said to be *Murray-von Neumann equivalent* if there exists a  $G$ -invariant element  $s \in \mathcal{B}(V, W) \otimes A$  such that  $s^* s = p$  and  $ss^* = q$ . Given a unitary finite dimensional representation  $\lambda: G \rightarrow \mathcal{U}(V)$  of  $G$  and a  $G$ -invariant projection  $p \in \mathcal{B}(V) \otimes A$ , and to emphasize role played by the representation  $\lambda$ , we denote the element in  $\mathcal{P}_G(A)$  it determines by  $(p, V, \lambda)$ . We let  $S_G(A)$  be the set of equivalence classes in  $\mathcal{P}_G(A)$  with addition given by direct sum.

We define the *equivariant  $K_0$ -group* of  $(G, A, \alpha)$ , denoted  $K_0^G(A)$ , to be the Grothendieck group of  $S_G(A)$ .

Define the *equivariant  $K_1$ -group* of  $(G, A, \alpha)$ , denoted  $K_1^G(A)$ , to be  $K_0^G(SA)$ , where the action of  $G$  on  $SA$  is trivial in the suspension direction.

If confusion is likely to arise as to with respect to what action the equivariant  $K$ -theory of  $A$  is being taken, we will write  $K_0^{\alpha}(A)$  and  $K_1^{\alpha}(A)$  instead of  $K_0^G(A)$  and  $K_1^G(A)$ .

**Remark 2.6.** The equivariant  $K$ -theory of  $(G, A, \alpha)$  is a module over the representation ring  $R(G)$  of  $G$ , which can be identified with  $K_0^G(\mathbb{C})$ , with the operation given by tensor product. This is, if  $(p, V, \lambda) \in \mathcal{P}_G(A)$  and  $(W, \mu)$  is a finite dimensional representation space of  $G$ , we define

$$(W, \mu) \cdot (p, V, \lambda) = (p \otimes 1_W, V \otimes W, \lambda \otimes \mu).$$

The induced operation  $R(G) \times K_0^G(A) \rightarrow K_0^G(A)$  makes  $K_0^G(A)$  into an  $R(G)$ -module. One defines the  $R(G)$ -module structure on  $K_1^G(A)$  analogously.

The following result is Julg's Theorem (Theorem 2.6.1 in [Phi87]).

**Theorem 2.7.** Let  $G$  be a compact group and let  $(G, A, \alpha)$  be a unital  $G$ -algebra. Then there are natural isomorphisms  $K_j^G(A) \cong K_j(A \rtimes_\alpha G)$  for  $j = 0, 1$ .

Let  $A$  and  $B$  be  $C^*$ -algebras, let  $G$  be a locally compact group, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  be continuous action of  $G$  on  $A$  and  $B$  respectively. We say that  $\alpha$  and  $\beta$  are *conjugate* if there exists an isomorphism  $\varphi: A \rightarrow B$  such that

$$\alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi$$

for all  $g$  in  $G$ . Isomorphisms of this form are called *equivariant*, and we usually use the notation  $\varphi: (A, \alpha) \rightarrow (B, \beta)$  to mean that  $\varphi$  satisfies the condition above.

A weaker form of equivalence for action is given by exterior equivalence, which we define below.

**Definition 2.8.** Let  $G$  be a locally compact group, let  $A$  and  $B$  be  $C^*$ -algebras, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  be continuous actions. We say that  $\alpha$  and  $\beta$  are *exterior equivalent* if there exist an isomorphism  $\theta: A \rightarrow B$  and a function  $u: G \rightarrow \mathcal{U}(\mathcal{M}(B))$  such that:

- (1)  $u_{gh} = u_g \theta(\alpha_g(\theta^{-1}(u_h)))$  for all  $g, h \in G$ ,
- (2) For each  $b \in B$ , the map  $G \rightarrow B$  given by  $g \mapsto u_g b$  is continuous.

The following result is folklore, and its proof is included here for the convenience of the reader.

**Proposition 2.9.** Let  $G$  be a locally compact abelian group, let  $A$  and  $B$  be  $C^*$ -algebras, and let  $\alpha: G \rightarrow \text{Aut}(A)$  and  $\beta: G \rightarrow \text{Aut}(B)$  be exterior equivalent actions. Then there exists an isomorphism  $\phi: A \rtimes_\alpha G \rightarrow A \rtimes_\beta G$  that intertwines the dual actions, this is, such that for every  $\chi \in \widehat{G}$ , we have  $\widehat{\beta}_\chi \circ \phi = \phi \circ \widehat{\alpha}_\chi$ .

*Proof.* Let  $\theta: A \rightarrow B$  and let  $u: G \rightarrow \mathcal{U}(\mathcal{M}(B))$  be as in the definition of exterior equivalence above. Define

$$\phi_0: L^1(G, A, \alpha) \rightarrow L^1(G, B, \beta)$$

by  $\phi_0(a)(g) = \theta(a(g))u_g^*$  for  $a \in L^1(G, A, \alpha)$  and  $g \in G$ . One readily checks that  $\phi_0$  is an isometric isomorphism of Banach  $*$ -algebras, and it therefore extends to an isomorphism  $\phi: A \rtimes_\alpha G \rightarrow B \rtimes_\beta G$  of  $C^*$ -algebras.

In order to check that  $\phi$  intertwines the dual actions, it is enough to check it on  $C_C(G, A)$ . (Note that  $\phi_0(C_C(G, A)) \subseteq C_C(G, B)$ .) Given a continuous function

$a \in C_C(G, A)$ , a group element  $g \in G$  and a character  $\chi \in \widehat{G}$ , we have

$$\begin{aligned} \phi(\widehat{\alpha}_\chi(a))(g) &= \theta(\widehat{\alpha}_\chi(a)(g)) u_g^* \\ &= \chi(g) \theta(a(g)) u_g^* \\ &= \chi(g) \phi(a)(g) \\ &= \widehat{\beta}_\chi(\phi(a))(g), \end{aligned}$$

and the proof follows.  $\square$

### 3. THE ROKHLIN PROPERTY FOR CIRCLE ACTIONS

In this section, we introduce the definition of the Rokhlin property for a circle action, and deduce some of the basic properties that will be needed to prove our classification results.

**Definition 3.1.** Let  $A$  be a unital  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be a continuous action. We say that  $\alpha$  has the *Rokhlin property* if for every  $\varepsilon > 0$  and every finite subset  $F \subseteq A$ , there exists a unitary  $u$  in  $A$  such that

- (1)  $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$  for all  $\zeta \in \mathbb{T}$ .
- (2)  $\|ua - au\| < \varepsilon$  for all  $a \in F$ .

**Remark 3.2.** Since compact subsets of metric spaces are completely bounded, one gets an equivalent notion if in Definition 3.1 above one allows the subset  $F$  of  $A$  to be norm compact instead of finite. We will make repeated use of this fact without mentioning it further.

The result below is an application of the fact that the action of  $\mathbb{T}$  on  $C(\mathbb{T})$  by left translation is equivariantly semiprojective. Informally speaking, this means that whenever we are given an almost equivariant unital homomorphism from  $C(\mathbb{T})$  into another unital  $C^*$ -algebra with a circle action, then there is a nearby *exactly equivariant* unital homomorphism from  $C(\mathbb{T})$  into the same algebra. We don't need to make this notion precise here, so we just present the result that we will use later. See [Phi12] for a rigorous treatment of the notion of equivariant semiprojectivity.

**Proposition 3.3.** Let  $A$  be a unital  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be a continuous action. Then  $\alpha$  has the Rokhlin property if and only if for every  $\varepsilon > 0$  and every finite subset  $F \subseteq A$ , there exists a unitary  $u$  in  $A$  such that

- (1)  $\alpha_\zeta(u) = \zeta u$  for all  $\zeta \in \mathbb{T}$ .
- (2)  $\|ua - au\| < \varepsilon$  for all  $a \in F$ .

The definition of the Rokhlin property differs from the conclusion of this proposition in that in condition (1), one only requires  $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$  for all  $\zeta \in \mathbb{T}$ .

*Proof.* The “if” implication is obvious. For the converse, let  $F \subseteq A$  be a finite subset and let  $\varepsilon > 0$ . Upon normalizing the finite set  $F$ , we may assume that  $\|a\| \leq 1$  for all  $a$  in  $F$ . Choose  $\varepsilon_0 < \frac{1}{3}$  small enough so that

$$\frac{4\varepsilon_0}{\sqrt{1-2\varepsilon_0}} + 3\varepsilon_0 < \varepsilon.$$

Choose a unitary  $v$  in  $A$  such that conditions (1) and (2) in Definition 3.1 are satisfied with  $\varepsilon_0$  in place of  $\varepsilon$ . Denote by  $\mu$  the normalized Haar measure on  $\mathbb{T}$ , and set

$$x = \int_{\mathbb{T}} \bar{\zeta} \alpha_\zeta(v) d\mu(\zeta).$$



Then  $\|x\| \leq 1$  and  $\|x - v\| \leq \varepsilon_0$ . One checks that  $\|x^*x - 1\| \leq 2\varepsilon_0 < 1$  and  $\|xx^* - 1\| < 1$ . It is also easy to verify that  $\alpha_\zeta(x) = \zeta x$  for all  $\zeta$  in  $\mathbb{T}$ , using translation invariance of  $\mu$ . Moreover,

$$\|(x^*x)^{-1}\| \leq \frac{1}{1 - \|1 - x^*x\|} \leq \frac{1}{1 - 2\varepsilon_0},$$

and thus  $\|(x^*x)^{-\frac{1}{2}}\| \leq \frac{1}{\sqrt{1 - 2\varepsilon_0}}$ .

Set  $u = x(x^*x)^{-\frac{1}{2}}$ . Then  $u$  is a unitary in  $A$ . Using that  $\|x\| \leq 1$  at the first step, and that  $0 \leq 1 - (x^*x)^{\frac{1}{2}} \leq 1 - x^*x$  at the third step, we get

$$\begin{aligned} \|u - x\| &\leq \|(x^*x)^{-\frac{1}{2}} - 1\| \\ &\leq \|(x^*x)^{-\frac{1}{2}}\| \|1 - (x^*x)^{\frac{1}{2}}\| \\ &\leq \frac{1}{\sqrt{1 - 2\varepsilon_0}} \|1 - x^*x\| \leq \frac{2\varepsilon_0}{\sqrt{1 - 2\varepsilon_0}}. \end{aligned}$$

We deduce that

$$\|u - v\| \leq \frac{2\varepsilon_0}{\sqrt{1 - 2\varepsilon_0}} + \varepsilon_0.$$

For  $\zeta$  in  $\mathbb{T}$ , we have  $\alpha_\zeta(x^*x) = x^*x$  and hence

$$\alpha_\zeta(u) = \alpha_\zeta\left(x(x^*x)^{-\frac{1}{2}}\right) = \zeta u,$$

so  $u$  satisfies condition (1) of the statement. Finally, if  $a \in F$  then

$$\begin{aligned} \|ua - au\| &\leq \|ua - va\| + \|va - av\| + \|av - au\| \\ &< \|u - v\| \|a\| + \varepsilon_0 + \|a\| \|v - u\| \\ &\leq \frac{4\varepsilon_0}{\sqrt{1 - 2\varepsilon_0}} + 3\varepsilon_0 < \varepsilon, \end{aligned}$$

as desired.  $\square$

If  $B$  is a  $C^*$ -algebra and  $\mathbb{Z}$  acts on  $B$ , then there is a continuous action of  $\mathbb{T}$  on  $B \rtimes \mathbb{Z}$ , namely, the dual action. Many relevant examples of circle actions arise in this way. Analogously, if  $\mathbb{T}$  acts on a  $C^*$ -algebra  $A$ , then there is a dual action of the integers on  $A \rtimes \mathbb{T}$ . It is therefore natural to ask what is the notion dual to the Rokhlin property, this is, characterize those integer actions that are dual or predual of a Rokhlin action of the circle. Propositions 3.6 and 3.7 answer this question: the dual notion is *approximate representability*, which we define below.

**Definition 3.4.** Let  $B$  be a  $C^*$ -algebra and let  $\beta$  be an automorphism of  $B$ . Then  $\beta$  is said to be *approximately representable* if there exists a unitary  $v \in (M(B)^\beta)_\infty$  such that  $\beta(b) = vbv^*$  for all  $b \in B$ .

It is relatively straightforward to characterize approximately representable automorphisms of unital separable  $C^*$ -algebras in terms of unitaries in the algebra rather than unitaries in the sequence algebra of its fixed point algebra.

**Lemma 3.5.** Let  $B$  be a separable unital  $C^*$ -algebra and let  $\beta$  be an automorphism of  $B$ . Then  $\beta$  is approximately representable if and only if for every  $\varepsilon > 0$  and every finite subset  $F \subseteq B$  there exists a unitary  $u \in \mathcal{U}(B)$  satisfying

$$(1) \quad \|\beta(b) - ubu^*\| < \varepsilon \text{ for all } b \in F, \text{ and}$$

$$(2) \quad \|\beta(u) - u\| < \varepsilon.$$

*Proof.* Assume that  $\beta$  is approximately representable. Let  $\varepsilon > 0$  and let  $F \subseteq B$  be a finite set. One can normalize  $F$  and assume that all its elements have norm at most 1. Using semiprojectivity of  $C(\mathbb{T})$ , let  $\delta > 0$  such that whenever  $w \in B$  satisfies  $\|w^*w - 1\| < \delta$  and  $\|ww^* - 1\| < \delta$ , then there exists a unitary  $\tilde{w}$  in  $B$  such that  $\|\tilde{w} - w\| < \frac{\varepsilon}{2}$ . Set  $\delta' = \min(\frac{\varepsilon}{4}, \delta)$ .

Use approximate representability of  $\beta$  to choose a unitary  $v = (v_n)_{n \in \mathbb{N}}$  in  $(B^\beta)_\infty$  such that  $\beta(b) = vbv^*$  for all  $b \in B$ . Since

- $\lim_{n \rightarrow \infty} \|\beta(b) - v_n b v_n^*\| = 0$  for all  $b \in B$
- $\lim_{n \rightarrow \infty} \|\beta(v_n) - v_n\| = 0$
- $\lim_{n \rightarrow \infty} \|v_n^* v_n - 1\| = 0$
- $\lim_{n \rightarrow \infty} \|v_n v_n^* - 1\| = 0$ ,

there exists  $n_0 \in \mathbb{N}$  such that all the quantities

$$\|\beta(b) - v_n b v_n^*\| \quad \|\beta(v_n) - v_n\| \quad \|v_n^* v_n - 1\| \quad \text{and} \quad \|v_n v_n^* - 1\|$$

are smaller than  $\delta'$  for all  $n \geq n_0$  and for all  $b \in F$ . Find a unitary  $v \in \mathcal{U}(B)$  such that  $\|u - v_{n_0}\| < \frac{\varepsilon}{2}$ . Then

$$\begin{aligned} \|\beta(b) - u b u^*\| &\leq \|\beta(b) - v_n b v_n^*\| + \|v_n b v_n^* - u b u^*\| \\ &< \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

for all  $b \in F$ . Likewise,  $\|\beta(u) - u\| < \varepsilon$ . Hence  $u$  is the desired unitary.

For the reverse implication, let  $(F_n)_{n \in \mathbb{N}}$  be an increasing family of finite subsets of  $B$  whose union is dense in  $B$ , and for each  $\varepsilon_n = \frac{1}{n}$ , let  $u_n$  be a unitary in  $B$  such that  $\|\beta(b) - u_n b u_n^*\| < \frac{1}{n}$  for every  $b \in F_n$  and  $\|\beta(u_n) - u_n\| < \varepsilon_n$ . Then  $v = \overline{(u_n)_{n \in \mathbb{N}}}$  is a unitary in  $(B^\beta)_\infty$  satisfying  $\beta(b) = v b v^*$  for all  $b \in B$ , using density of  $\bigcup_{n \in \mathbb{N}} F_n$  in  $B$ .  $\square$

The following proposition characterizes integer actions whose dual action has the Rokhlin property.

**Proposition 3.6.** Let  $B$  be a separable unital  $C^*$ -algebra and let  $\beta \in \text{Aut}(B)$ . Consider the dual action  $\hat{\beta}: \mathbb{T} \rightarrow \text{Aut}(B \rtimes_\beta \mathbb{Z})$ . Then  $\beta$  is approximately representable if and only if  $\hat{\beta}$  has the Rokhlin property.

*Proof.* Assume that  $\beta$  is approximately representable. Let  $F$  be a finite subset of  $B \rtimes_\beta \mathbb{Z}$  and let  $\varepsilon > 0$ . Denote by  $u$  the canonical unitary in the crossed product. Since  $B$  and  $u$  generate  $B \rtimes_\beta \mathbb{Z}$ , one can assume that  $F = F' \cup \{u\}$ , where  $F'$  is a finite subset of  $B$ . Use Lemma 3.5 to choose a unitary  $v$  in  $B$  satisfying

- (1)  $\|\beta(b) - v b v^*\| < \varepsilon$  for all  $b \in F'$ , and
- (2)  $\|\beta(v) - v\| < \varepsilon$ .

Since  $\beta(b) = u b u^*$  for every  $b \in B$ , if we let  $w = v^* u$ , the first of these conditions is equivalent to  $\|w b - b w\| < \varepsilon$  for all  $b \in F'$ , while the second one is equivalent to  $\|w u - u w\| < \varepsilon$ . On the other hand,

$$\hat{\beta}_\zeta(w) = \hat{\beta}_\zeta(v^* u) = v^*(\zeta u) = \zeta w$$

for all  $\zeta$  in  $\mathbb{T}$ . Thus,  $w$  is the desired unitary, and  $\widehat{\beta}$  has the Rokhlin property.

Conversely, assume that  $\widehat{\beta}$  has the Rokhlin property. Let  $F' \subseteq B$  be a finite subset and let  $\varepsilon > 0$ . Set  $F = F' \cup \{u\}$  and, using Proposition 3.3, choose  $w$  such that  $\|wb - bw\| < \varepsilon$  for all  $b \in F$ , and  $\widehat{\beta}(w) = \zeta w$  for all  $\zeta \in \mathbb{T}$ . Set  $v = uw^*$ . Then  $v$  belongs to  $B$  since

$$\widehat{\beta}_\zeta(v) = \zeta u \bar{\zeta} w^* = uw^* = v$$

for all  $\zeta$  in  $\mathbb{T}$  and  $(B \rtimes_\beta \mathbb{Z})^\mathbb{T} = B$ . On the other hand,

$$\|\beta(b) - v b v^*\| = \|u b u^* - v b v^*\| = \|(v^* u) b - b (v^* u)\| = \|wb - bw\| < \varepsilon,$$

for all  $b \in F'$ , and

$$\|\beta(v) - v\| = \|u(uw^*)u^* - uw^*\| = \|w^*u^* - u^*w^*\| = \|uw - wu\| < \varepsilon.$$

Hence  $v$  is an implementing unitary for  $F'$  and  $\varepsilon$ , and thus  $\beta$  is approximately representable by Lemma 3.5.  $\square$

The following proposition characterizes integer actions dual to circle actions with the Rokhlin property.

**Proposition 3.7.** Let  $A$  be a separable unital  $C^*$ -algebra, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be a continuous action. Denote by  $\widehat{\alpha}: \mathbb{Z} \rightarrow \text{Aut}(A \rtimes_\alpha \mathbb{T})$  the dual action of  $\alpha$ . Then  $\alpha$  has the Rokhlin property if and only if  $\widehat{\alpha}$  is approximately representable.

*Proof.* Assume that  $\alpha$  has the Rokhlin property. We claim that there is an equivariant unital embedding  $C(\mathbb{T}) \hookrightarrow A_\infty \cap A'$ , the action on  $C(\mathbb{T})$  being translation, and the (not necessarily continuous) action on  $A_\infty \cap A'$  being  $\alpha_\infty$ .

Choose an increasing family  $(F_n)_{n \in \mathbb{N}}$  of finite subsets of  $A$  whose union is dense in  $A$ . Using the Rokhlin property for  $\alpha$ , for every  $n \in \mathbb{N}$  choose a unitary  $u_n \in \mathcal{U}(A)$  such that

- (1)  $\|u_n a - a u_n\| < \frac{1}{n}$  for all  $a \in F_n$ , and
- (2)  $\|\alpha_\zeta(u_n) - \zeta u_n\| < \frac{1}{n}$  for all  $\zeta \in \mathbb{T}$ .

Set  $u = (u_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$  and denote by  $\bar{u}$  its image in  $A_\infty$  under the canonical surjection  $\ell^\infty(\mathbb{N}, A) \rightarrow A_\infty$ . Then  $\bar{u}$  is a unitary in  $A_\infty$ , and it is easy to check that it belongs to the relative commutant of  $A$  in  $A_\infty$ . We therefore get a unital homomorphism  $\varphi: C(\mathbb{T}) \rightarrow A_\infty \cap A'$  which is readily seen to be equivariant. Since the only invariant ideals of  $C(\mathbb{T})$  are  $\{0\}$  and  $C(\mathbb{T})$  and  $\varphi$  is unital, it follows that it is injective. This is the desired embedding.

Denote by  $\omega \in A_\infty = (M(A \rtimes_\alpha \mathbb{T})^\alpha)_\infty$  the image under the embedding  $\varphi$  of the canonical generator  $z \in C(\mathbb{T})$  given by  $z(\zeta) = \zeta$  for  $\zeta \in \mathbb{T}$ , and denote by  $\lambda$  the implementing unitary representation of  $\mathbb{T}$  in  $M(A \rtimes_\alpha \mathbb{T})$  for  $\alpha$ . In  $(M(A \rtimes_\alpha \mathbb{T}))_\infty$ , we have

$$\omega^* \lambda_\zeta \omega = \zeta \lambda_\zeta \quad \text{and} \quad \omega a = a \omega$$

for all  $\zeta \in \mathbb{T}$  and for all  $a \in A$ . Therefore, if  $\alpha$  has the Rokhlin property, then  $\widehat{\alpha}$  is implemented by  $\omega^*$ , and thus it is approximately representable. The converse follows from the same computation, since we have  $M(A \rtimes_\alpha \mathbb{T})^\alpha = A$ .  $\square$

Even when the algebra is not separable, we still have the following result which will be enough to deduce some important facts about actions with the Rokhlin property.

**Lemma 3.8.** Let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property on a unital  $C^*$ -algebra  $A$ . Then the dual automorphism of  $\alpha$  is approximately inner.

*Proof.* It is enough to show that the extension of  $\widehat{\alpha}$  to  $M(A \rtimes_{\alpha} \mathbb{T})$  is approximately inner. As in the proof of the proposition above, denote by  $\lambda: \mathbb{T} \rightarrow \mathcal{U}(M(A \rtimes_{\alpha} \mathbb{T}))$  the implementing unitary representation. Let  $\varepsilon > 0$  and let  $F \subseteq M(A \rtimes_{\alpha} \mathbb{T})$  be a finite set. We can assume that there are  $n \in \mathbb{N}$ , circle elements  $\zeta_1, \dots, \zeta_n \in \mathbb{T}$  and a finite subset  $F' \subseteq A$  such that  $F = F' \cup \{\lambda_{\zeta_1}, \dots, \lambda_{\zeta_n}\}$ . We can moreover assume (by increasing  $n$  and approximating the unitaries  $\lambda_{\zeta_j}$ ) that  $\lambda_{\zeta_j} = \lambda_{\zeta_1}^j$  for  $j = 1, \dots, n$ . Use Proposition 3.3 to choose a unitary  $u$  in  $A$  such that  $\|ua - au\| < \varepsilon$  for all  $a \in F'$  and such that, in particular,  $\alpha_{\zeta_1}(u) = \zeta_1 u$ . A similar computation as in the proof of Proposition 3.7 then shows that  $u$  approximately implements the automorphism  $\widehat{\alpha}$  of  $M(A \rtimes_{\alpha} \mathbb{T})$  on the set  $F$ .  $\square$

We refer the reader to [Phi09] for the definitions and basic properties of the Connes spectrum and hereditary saturation. Since we will not use these concepts beyond the following remark, we do not include here the relevant definitions.

**Remark 3.9.** Let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action of the circle on a unital  $C^*$ -algebra  $A$  with the Rokhlin property. By Theorem 6.14 of [Phi09] and Corollary 3.8, it follows that  $\alpha$  has full Connes spectrum, this is,  $\widetilde{\Gamma}(\alpha) = \mathbb{Z}$ . Moreover,  $\alpha$  is hereditarily saturated by Corollary 6.10 of [Phi09]. In particular,  $A^{\alpha}$  and  $A \rtimes_{\alpha} \mathbb{T}$  are Morita equivalent, and if  $A$  is separable, it follows that  $K_*(A^{\alpha}) \cong K_*(A \rtimes_{\alpha} \mathbb{T})$ .

In this context, the isomorphism  $K_*(A^{\alpha}) \cong K_*(A \rtimes_{\alpha} \mathbb{T})$  will be shown to hold even if  $A$  is not separable. Indeed, a much stronger statement is true, namely  $A \rtimes_{\alpha} \mathbb{T}$  is isomorphic to  $A^{\alpha} \otimes \mathcal{K}(L^2(\mathbb{T}))$ . See Corollary 3.12.

Lemma 3.8 has strong implications on the  $K$ -theory of an algebra that admits a circle action with the Rokhlin property. Indeed, assume that  $A$  is a unital  $C^*$ -algebra and that  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  is an action with the Rokhlin property. The Pimsner-Voiculescu for  $\alpha$  (see Subsection 10.6 in [Bla98]) is

$$\begin{array}{ccccc}
 K_0(A \rtimes_{\alpha} \mathbb{T}) & \xrightarrow{1-K_0(\widehat{\alpha})} & K_0(A \rtimes_{\alpha} \mathbb{T}) & \xrightarrow{\quad} & K_0(A) \\
 \uparrow & & & & \downarrow \\
 K_1(A) & \xleftarrow{\quad} & K_1(A \rtimes_{\alpha} \mathbb{T}) & \xleftarrow{1-K_1(\widehat{\alpha})} & K_1(A \rtimes_{\alpha} \mathbb{T}).
 \end{array}$$

The automorphism  $K_j(\widehat{\alpha})$  is the identity on  $K_j(A \rtimes_{\alpha} \mathbb{T})$  for  $j = 0, 1$  by Lemma 3.8. Thus the exact sequence above splits into two short exact sequences

$$0 \rightarrow K_j(A \rtimes_{\alpha} \mathbb{T}) \rightarrow K_j(A) \rightarrow K_{1-j}(A \rtimes_{\alpha} \mathbb{T}) \rightarrow 0$$

for  $j = 0, 1$ . It follows that if one of the  $K$ -groups of  $A$  is trivial, then so is the other one. In particular, there are no circle actions with the Rokhlin property on AF-algebras, AI-algebras, the Jiang-Su algebra  $\mathcal{Z}$ , or any of the Cuntz algebras  $\mathcal{O}_n$  with  $n > 2$ . On the other hand, there are many such actions on  $\mathcal{O}_2$ : it is shown in Theorem 6.8 of [Gar14a] that circle actions with the Rokhlin property on  $\mathcal{O}_2$  are generic.

There are other restrictions that follow from the short exact sequences above. We list a few of them:

- $K_0(A)$  is finitely generated if and only if  $K_1(A)$  is finitely generated.
- $K_0(A)$  is torsion if and only if  $K_1(A)$  is torsion.
- If  $K_0(A)$  and  $K_1(A)$  are free, then  $K_0(A) \cong K_1(A)$ .

- More generally, the free ranks of the  $K$ -groups of  $A$  must agree, this is,  $\text{rk}K_0(A) = \text{rk}K_1(A)$ .

It is nevertheless not clear at this point whether  $K_0(A) = \mathbb{Z}$  and  $K_1(A) = \mathbb{Z} \oplus \mathbb{Z}_2$  can happen. This combination of  $K$ -groups will be ruled out by Theorem 5.5.

It is a standard fact (see Theorem 3.1 in [Kis81]) that formation of (reduced) crossed products by pointwise outer actions of discrete groups preserves simplicity. However, the corresponding statement for not necessarily discrete groups is false, even in the compact case. For example, consider the gauge action  $\gamma$  of  $\mathbb{T}$  on  $\mathcal{O}_\infty$ , which is given by  $\gamma_\zeta(s_j) = \zeta s_j$  for all  $\zeta$  in  $\mathbb{T}$  and all  $j \in \mathbb{N}$ . Then  $\gamma$  is pointwise outer by the Theorem in [MT93], but the crossed product is well-known to be non-simple.

Intuitively speaking, sufficiently outer actions of, say, compact groups, ought to preserve simplicity. (And this should be a test for what “sufficiently outer” means for a compact group action.) Since the Rokhlin property is a rather strong form of outerness, the next result should come as no surprise.

**Proposition 3.10.** Let  $A$  be a simple unital  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then  $A^\alpha$  and  $A \rtimes_\alpha \mathbb{T}$  are simple  $C^*$ -algebras.

*Proof.* Since  $A^\alpha$  and  $A \rtimes_\alpha \mathbb{T}$  are Morita equivalent by Remark 3.9, it is enough to show that  $A \rtimes_\alpha \mathbb{T}$  is simple. Let  $I$  be an ideal of  $A \rtimes_\alpha \mathbb{T}$ . Since  $\widehat{\alpha}$  is approximately inner by Lemma 3.8, it follows that  $\widehat{\alpha}(I) \subseteq I$ . Hence the crossed product  $I \rtimes_{\widehat{\alpha}} \mathbb{Z}$  of the restriction of  $\widehat{\alpha}$  to  $I$  is an ideal in  $(A \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ . Now, there is a natural isomorphism  $(A \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z} \cong A \otimes \mathcal{K}$  by Takai duality (see Theorem 7.9.3 in [Ped79]). Since  $A$  is simple, it follows that  $I \rtimes_{\widehat{\alpha}} \mathbb{Z}$  is isomorphic to either  $A \otimes \mathcal{K}$  or to  $\{0\}$ . By Lemma 2.8.2 in [Phi87], there is an isomorphism

$$((A \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}) / (I \rtimes_{\widehat{\alpha}} \mathbb{Z}) \cong (A \rtimes_\alpha \mathbb{T} / I) \rtimes_{\widehat{\alpha}} \mathbb{Z}.$$

It follows that  $I \rtimes_{\widehat{\alpha}} \mathbb{Z} = A \otimes \mathcal{K}$  if and only if  $I = A \rtimes_\alpha \mathbb{T}$  and that  $I \rtimes_{\widehat{\alpha}} \mathbb{Z} = \{0\}$  if and only if  $I = \{0\}$ . Thus,  $A \rtimes_\alpha \mathbb{T}$  is simple.  $\square$

We point out that a similar argument, but with somewhat more work, shows that if we drop the assumption that  $A$  is simple, then every ideal in  $A \rtimes_\alpha \mathbb{T}$  has the form  $I \rtimes_\alpha \mathbb{T}$  for some  $\alpha$ -invariant ideal  $I$  in  $A$ . The proof depends on the concrete picture of the isomorphism  $(A \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z} \cong A \otimes \mathcal{K}(L^2(\mathbb{T}))$  given by Takai duality. Since we deal mostly with simple  $C^*$ -algebras in the present work, we do not prove the more general result here.

We close this section by showing that every circle action with the Rokhlin property arises as a dual action. Our proof uses the machinery developed by Ruy Exel for partial actions together with the fact that the action of  $\mathbb{T}$  on  $C(\mathbb{T})$  by left translation is equivariantly semiprojective (see Proposition 3.3).

**Theorem 3.11.** Let  $A$  be a  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then there exist an automorphism  $\theta$  of  $A^\alpha$  and an equivariant isomorphism

$$\varphi: (A^\alpha \rtimes_\theta \mathbb{Z}, \widehat{\theta}) \rightarrow (A, \alpha)$$

which is the identity on  $A^\alpha$ .

Moreover,  $\theta$  is unique up to unitary equivalence. This is, if  $\theta'$  is another automorphism of  $A^\alpha$  and  $\varphi': (A^\alpha \rtimes_{\theta'} \mathbb{Z}, \widehat{\theta'}) \rightarrow (A, \alpha)$  is another equivariant isomorphism, then there is a unitary  $w$  in  $A^\alpha$  such that  $\theta = \text{Ad}(w) \circ \theta'$ .

*Proof.* We claim that  $\alpha$  is semisaturated and regular (see Definition 2.2).

We begin by checking that  $\alpha$  is semisaturated. Let  $n \in \mathbb{Z}$  and let  $a \in A_n$ . Then  $\alpha_\zeta(a) = \zeta^n a$  for all  $\zeta \in \mathbb{T}$ . Using Proposition 3.3, choose a unitary  $u \in \mathcal{U}(A)$  such that  $\alpha_\zeta(u) = \zeta u$  for all  $\zeta \in \mathbb{T}$ . Then  $au^{-n}$  belongs to  $A^\alpha$ , and thus  $a = au^{-n}u^n$  is in  $A^\alpha A_1^n$ . It follows that  $A = \bigoplus_{n \in \mathbb{Z}} A_n$  is generated as a  $C^*$ -algebra by  $A^\alpha$  and  $A_1$ , and hence  $\alpha$  is semisaturated.

We now check that  $\alpha$  is regular. With  $u \in \mathcal{U}(A)$  as above, define

$$\theta: A_1^* A_1 \rightarrow A_1 A_1^* \quad \text{by} \quad \theta(x) = uxu^*$$

for  $x$  in  $A_1^* A_1$ , and define

$$\lambda: A_1^* \rightarrow A_1 A_1^* \quad \text{by} \quad \lambda(y) = uy$$

for  $y$  in  $A_1^*$ . It is clear that  $\theta$  is an isomorphism and that  $\lambda$  is a linear isometry. Moreover, for  $x, y \in A_1$ , for  $a \in A_1^* A_1$ , and for  $b \in A_1 A_1^*$ , we have

$$\begin{aligned} \lambda(x^* b) &= ux^* b = \lambda(x^*) b \\ \lambda(ax^*) &= uax^* = uau^* ux^* = \theta(a) \lambda(x^*) \\ \lambda(x^*)^* \lambda(y^*) &= (ux^*)^* (uy^*) = xy^* \\ \lambda(x^*) \lambda(y^*)^* &= ux^* (uy^*)^* = ux^* yu^* = \theta(x^* y). \end{aligned}$$

Hence  $\alpha$  is regular. This proves the claim.

Since  $u$  belongs to  $A_1$ , it follows that  $1 = uu^* = u^*u$  belongs to both  $A_1 A_1^*$  and  $A_1^* A_1$ , so these ideals are equal to  $A^\alpha$ . Now, Theorem 4.21 in [Exe94] (here reproduced as Theorem 2.4) and Remark 2.3 finish the proof of the first claim.

We now turn to uniqueness of  $\theta$ . Let  $\theta'$  and  $\varphi'$  be as in the statement. Let  $v$  be the canonical unitary in  $A^\alpha \rtimes_\theta \mathbb{Z}$  that implements  $\widehat{\theta}$ , and let  $v'$  be the canonical unitary in  $A^\alpha \rtimes_{\theta'} \mathbb{Z}$  that implements  $\widehat{\theta'}$ . Set  $w = \varphi(v) \varphi'(v')^*$ , which is a unitary in  $A$ . We claim that  $w$  is fixed by  $\alpha$ . Indeed, for  $\zeta \in \mathbb{T}$ , we use the facts that  $\varphi$  and  $\varphi'$  are equivariant, to obtain

$$\alpha_\zeta(w) = \varphi(\widehat{\theta}_\zeta(v)) \varphi'(\widehat{\theta'_\zeta}(v'))^* = \zeta \varphi(v) \bar{\zeta} \varphi'(v') = w.$$

Whence  $w$  belongs to  $A^\alpha$ . Finally, given  $a$  in  $A^\alpha$ , we have

$$\begin{aligned} (\text{Ad}(w) \circ \theta)(a) &= (\varphi(v) \varphi'(v')^*) (\varphi'(v') a \varphi'(v')^*) (\varphi'(v') \varphi(v)^*) \\ &= \varphi(v) a \varphi(v)^* = \theta'(a), \end{aligned}$$

and the result follows.  $\square$

If  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  is an action of the circle on a unital  $C^*$ -algebra  $A$  with the Rokhlin property, then we will usually denote an automorphism of  $A^\alpha$  as in the conclusion of Theorem 3.11, which is unique up to unitary equivalence, by  $\check{\alpha}$ . Since  $\widehat{\check{\alpha}}$  is conjugate to  $\alpha$ , we will usually refer to  $\check{\alpha}$  as the *predual* automorphism of  $\alpha$  (hence the notation  $\check{\alpha}$ ).

**Corollary 3.12.** Let  $A$  be a  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then there is a natural isomorphism  $A \rtimes_\alpha \mathbb{T} \cong A^\alpha \otimes \mathcal{K}(L^2(\mathbb{T}))$ .

*Proof.* This is an immediate consequence of Theorem 3.11 together with the natural isomorphism given by Takai duality.  $\square$

#### 4. AN AVERAGING TECHNIQUE

The goal of this section is to develop an averaging technique using the Rokhlin property that will allow us to define, given a compact subset  $F$  of  $A$ , a linear map  $A \rightarrow A^\alpha$  which is an approximate  $*$ -homomorphism on  $F$ , in a suitable sense. See Theorem 4.3.

Some results in this section will be stated and proved in greater generality than needed here because the extra flexibility in the statements (particularly that of Theorem 4.3) will be crucial in the proof of Theorem 3.22 in [Gar14b], where we will construct homotopies between linear maps  $A \rightarrow A^\alpha$  coming from different choices of tolerance and compact set. In this paper, Theorem 4.3 will be used to prove Proposition 5.1, Theorem 5.2 and Theorem 6.3.

We begin with some fairly general observations.

Let  $G$  be a compact group, let  $A$  be a unital  $C^*$ -algebra, and let  $\alpha: G \rightarrow \text{Aut}(A)$  be a continuous action. Identify  $C(G) \otimes A$  with  $C(G, A)$ , and denote by  $\gamma: G \rightarrow \text{Aut}(C(G, A))$  the diagonal action, this is,  $\gamma_g(a)(h) = \alpha_g(a(g^{-1}h))$  for all  $g, h \in G$  and all  $a \in C(G, A)$ . Define an *averaging process*  $\phi: C(G, A) \rightarrow C(G, A)$  by

$$\phi(a)(g) = \alpha_g(a(1))$$

for all  $a$  in  $C(G, A)$  and all  $g$  in  $G$ .

We specialize to  $G = \mathbb{T}$  now. Let  $\varepsilon > 0$  and let  $F$  be a compact subset of  $C(\mathbb{T}, A)$ . Set

$$F' = \bigcup_{\zeta \in \mathbb{T}} \gamma_\zeta(F) \quad \text{and} \quad F'' = \{a(\zeta): a \in F', \zeta \in \mathbb{T}\},$$

which are compact subsets of  $C(\mathbb{T}, A)$  and  $A$ , respectively. Choose  $\delta > 0$  such that whenever  $\zeta_1$  and  $\zeta_2$  in  $\mathbb{T}$  satisfy  $|\zeta_1 - \zeta_2| < \delta$ , then

$$\|\alpha_{\zeta_1}(a(1)) - \alpha_{\zeta_2}(a(1))\| < \frac{\varepsilon}{2}$$

for all  $a$  in  $F'$ . Let  $(f_j)_{j=0}^n$  be a partition of unity of  $\mathbb{T}$  and let  $\zeta_1, \dots, \zeta_n$  be group elements in  $\mathbb{T}$  such that  $f_j(\zeta) \neq 0$  for some  $\zeta$  in  $\mathbb{T}$  implies  $|\zeta - \zeta_j| < \frac{\delta}{2}$ . In particular, this implies that  $|\zeta_j - \zeta_k| < \delta$  whenever the supports of  $f_j$  and  $f_k$  are not disjoint. (Such partition of unity and group elements are easy to construct: take, for example, the support of each function  $f_j$  to be an interval of radius  $\frac{\delta}{2}$ , and let  $\zeta_j$  be the center of its support.)

For use in the proof of the following lemma, we recall the following standard fact about self-adjoint elements in a  $C^*$ -algebra: if  $A$  is a unital  $C^*$ -algebra and  $a, b \in A$  with  $b^* = b$ , then  $-\|b\|a^*a \leq a^*ba \leq \|b\|a^*a$ , and hence  $\|a^*ba\| \leq \|b\|\|a\|^2$ .

**Lemma 4.1.** Adopt the notation and assumptions of the discussion above. Then  $\phi$  is a homomorphism and its range is contained in the fixed point subalgebra of  $C(\mathbb{T}, A)$ . Moreover, for every  $\zeta \in \mathbb{T}$  and every  $a$  in  $F'$ , we have

$$\left\| \gamma_\zeta \left( \sum_{j=1}^n f_j \alpha_{\zeta_j}(a(1)) \right) - \sum_{j=1}^n f_j \alpha_{\zeta_j}(a(1)) \right\| < \varepsilon.$$

*Proof.* We begin by showing that the averaging process  $\phi: C(\mathbb{T}, A) \rightarrow C(\mathbb{T}, A)$  is a homomorphism. Let  $a, b \in C(\mathbb{T}, A)$ , and let  $\zeta$  in  $\mathbb{T}$ . We have

$$(\phi(a)\phi(b))(\zeta) = \alpha_\zeta(a(1))\alpha_\zeta(b(1)) = \alpha_\zeta(ab(1)) = \phi(ab)(\zeta),$$

showing that  $\phi$  is multiplicative. It is clearly linear and preserves the involution, so it is a homomorphism.

We will now show that  $\gamma_\lambda(\phi(a)) = \phi(a)$  for all  $\lambda$  in  $\mathbb{T}$  and all  $a$  in  $C(\mathbb{T}, A)$ . Indeed, for  $\zeta$  in  $\mathbb{T}$ , we have

$$\gamma_\lambda(\phi(a))(\zeta) = \alpha_\lambda(\phi(a)(\lambda^{-1}\zeta)) = \alpha_\lambda(\alpha_{\lambda^{-1}\zeta}(a(1))) = \alpha_\zeta(a(1)) = \phi(a)(\zeta),$$

which proves the claim.

Since every element in a  $C^*$ -algebra is the linear combination of two self-adjoint elements, we may assume without loss of generality that every element of  $F$  is self-adjoint, so that the same holds for the elements of  $F'$  and  $F''$ . Given  $\zeta$  in  $\mathbb{T}$  and  $a$  in  $F'$ , we have

$$\begin{aligned} \phi(a)(\zeta) - \sum_{j=1}^n f_j(\zeta)\alpha_{\zeta_j}(a(1)) &= \sum_{j=1}^n f_j(\zeta)^{1/2}(\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1)))f_j(\zeta)^{1/2} \\ &\leq \sum_{j=1}^n \|\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1))\|f_j(\zeta). \end{aligned}$$

Now, for  $j = 1, \dots, n$ , if  $f_j(\zeta) \neq 0$ , then  $|\zeta_j - \zeta| < \delta$ , and hence  $\|\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1))\| < \frac{\varepsilon}{2}$ . In particular, we conclude that

$$-\frac{\varepsilon}{2} < \phi(a)(\zeta) - \sum_{j=1}^n f_j(\zeta)\alpha_{\zeta_j}(a(1)) < \frac{\varepsilon}{2}.$$

Since  $\zeta$  is arbitrary, we deduce that  $\left\|\phi(a) - \sum_{j=1}^n f_j\alpha_{\zeta_j}(a(1))\right\| < \frac{\varepsilon}{2}$ . Since  $\phi(a)$  is fixed by the action  $\gamma$ , the result follows from an easy application of triangle inequality.  $\square$

If we start with a compact subset of  $A$ , viewed as a compact subset of  $C(\mathbb{T}, A)$  consisting of constant functions, then the above lemma shows that any partition of unity with sufficiently small supports provides us with a way to take a discrete average over the group  $\mathbb{T}$ . We will later see that this discrete averaging has the advantage of being almost multiplicative in an appropriate sense. See Theorem 4.3.

We come back to actions with the Rokhlin property in the next lemma.

**Lemma 4.2.** Let  $A$  be a unital  $C^*$ -algebra, let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property, let  $\varepsilon > 0$ , let  $F$  be a compact subset of  $A$  and let  $S$  be a compact subset of  $C(\mathbb{T})$  consisting of positive functions. Then there exists a unitary  $u$  in  $A$  such that  $\alpha_\zeta(u) = \zeta u$  for all  $\zeta$  in  $\mathbb{T}$  and

$$\|af(u) - f(u)a\| < \frac{\varepsilon}{|S|}$$

for all  $a$  in  $F$  and all  $f$  in  $S$ .

*Proof.* For every  $m$  in  $\mathbb{N}$ , use Proposition 3.3 for  $\alpha$  to find a unitary  $u_m$  in  $A$  such that

- $\alpha_\zeta(u_m) = \zeta u_m$  for all  $\zeta \in \mathbb{T}$ .
- $\|u_m a - a u_m\| < \frac{1}{m}$  for all  $a \in F$ .



It is clear that

$$\lim_{m \rightarrow \infty} \|af(u_m) - f(u_m)a\| = 0$$

for all  $a$  in  $F$  and all  $f$  in  $C(\mathbb{T})$ . Since  $S$  is compact, one can choose  $m$  large enough so that, with  $u = u_m$ , we have

$$\|af(u) - f(u)a\| < \frac{\varepsilon}{|S|}$$

for all  $a$  in  $F$  and all  $f$  in  $S$ , as desired.  $\square$

Let  $A$  be a  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be a continuous action. We denote by  $E: A \rightarrow A^\alpha$  the standard conditional expectation. If  $\mu$  denotes the normalized Haar measure on  $\mathbb{T}$ , then  $E$  is given by

$$E(a) = \int_{\mathbb{T}} \alpha_\zeta(a) d\mu(\zeta)$$

for all  $a$  in  $A$ .

The way the next theorem is formulated will be convenient in the proof of a result in [Gar14b]. See Remark 4.4 below.

**Theorem 4.3.** Let  $A$  be a unital  $C^*$ -algebra, let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property, let  $\varepsilon > 0$  and let  $F$  be a compact subset of  $A$ . Set  $F' = \bigcup_{\zeta \in \mathbb{T}} \alpha_\zeta(F)$ , which is a compact subset  $A$ . Let  $\delta > 0$  such that whenever  $\zeta_1$  and  $\zeta_2$  in  $\mathbb{T}$  satisfy  $|\zeta_1 - \zeta_2| < \delta$ , then

$$\|\alpha_{\zeta_1}(a) - \alpha_{\zeta_2}(a)\| < \frac{\varepsilon}{2}$$

for all  $a$  in  $F'$ . Let  $(f_j)_{j=0}^n$  be a partition of unity of  $\mathbb{T}$  and let  $\zeta_1, \dots, \zeta_n$  be group elements in  $\mathbb{T}$  such that  $f_j(\zeta) \neq 0$  for some  $\zeta$  in  $\mathbb{T}$  implies  $|\zeta - \zeta_j| < \frac{\delta}{2}$ . Let  $u$  be a unitary as in the conclusion of Lemma 4.2 for the tolerance  $\varepsilon$ , compact subset  $F' \subseteq A$  and compact subset  $S = \{f_j, f_j^{1/2}: j = 1, \dots, n\} \subseteq C(\mathbb{T})$ .

Define a linear map  $\sigma: A \rightarrow A^\alpha$  by

$$\sigma(a) = E \left( \sum_{j=1}^n f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \right)$$

for all  $a$  in  $A$ . Then  $\sigma$  is unital and completely positive, and

$$\|\sigma(ab) - \sigma(a)\sigma(b)\| < \varepsilon(\|a\| + \|b\| + 3)$$

for all  $a$  and  $b$  in  $F'$ .

We point out that one can always choose a positive number  $\delta > 0$ , a partition of unity  $(f_j)_{j=1}^n$  and a unitary  $u$  in  $A$  satisfying the hypotheses of this theorem. We will use this fact without further reference in the future, and we will simply say “choose  $\delta, (f_j)_{j=1}^n$  and  $u$  as in the assumptions of Theorem 4.3”.

*Proof.* It is clear that  $\sigma$  is linear, completely positive, and unital. In particular, it is completely contractive.

We claim that

$$\left\| \sigma(a) - \sum_{j=1}^n f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \right\| \leq \varepsilon$$

for all  $a$  in  $F'$ .

Denote by  $\psi: C(\mathbb{T}, A) \rightarrow A$  the equivariant completely positive contractive map which is the identity on  $A$ , and sends the canonical generator of  $C(\mathbb{T})$  to  $u$ . Let  $a$  be in  $F'$ . For  $\zeta$  in  $\mathbb{T}$ , we use Lemma 4.1 at the last step to show that

$$\begin{aligned}
& \left\| \alpha_\zeta \left( \sum_{j=1}^n f_j(u_m)^{1/2} \alpha_{\zeta_j}(a) f_j(u_m)^{1/2} \right) - \sum_{j=1}^n f_j(u_m)^{1/2} \alpha_{\zeta_j}(a) f_j(u_m)^{1/2} \right\| \\
&= \left\| \alpha_\zeta \left( \psi \left( \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right) \right) - \psi \left( \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right) \right\| \\
&= \left\| \psi \left( \gamma_\zeta \left( \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right) - \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right) \right\| \\
&\leq \left\| \gamma_\zeta \left( \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right) - \sum_{j=1}^n f_j \otimes \alpha_{\zeta_j}(a) \right\| < \varepsilon,
\end{aligned}$$

as desired.

Note that  $\|[f_j(u)^{1/2}, a]\| < \frac{\varepsilon}{2n}$  for all  $a$  in  $F'$ . Let  $a$  and  $b$  in  $F'$ . Using at the third step that  $f_j f_k \neq 0$  implies  $|\zeta_j - \zeta_k| < \delta$ , we have

$$\begin{aligned}
\sigma(a)\sigma(b) &\approx_{\varepsilon(\|a\|+\|b\|)} \sum_{j,k=1}^n f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} f_k(u)^{1/2} \alpha_{\zeta_k}(b) f_k(u)^{1/2} \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \alpha_{\zeta_k}(b) f_k(u) \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \alpha_{\zeta_j}(b) f_k(u) \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} f_k(u) \\
&= \sum_{j=0}^n f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} \left( \sum_{k: f_k f_j \neq 0} f_k(u) \right) \\
&= \sum_{j=0}^n f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} = \sigma(ab).
\end{aligned}$$

Hence  $\|\sigma(a)\sigma(b) - \sigma(ab)\| < \varepsilon(\|a\| + \|b\| + 3)$ , as desired.  $\square$

**Remark 4.4.** For the immediate applications of Theorem 4.3, it will be enough to choose any  $\delta, (f_j)_{j=1}^n$  and  $u$  satisfying the assumptions of said theorem. However, we will need the more general statement in the proof of Theorem 3.22 in [Gar14b], in which we will need to construct homotopies between the linear maps  $\sigma$  obtained from different choices of  $\delta, (f_j)_{j=1}^n$  and  $u$ .

## 5. $K$ -THEORETIC OBSTRUCTIONS TO THE ROKHLIN PROPERTY

In this section, we show that there are severe obstructions on the  $K$ -theory of a unital  $C^*$ -algebra that admits a circle action with the Rokhlin property; see

Theorem 5.5. We will also see that the canonical embedding  $A^\alpha \rightarrow A$  induces injective group homomorphisms  $K_*(\iota): K_*(A^\alpha) \rightarrow K_*(A)$ , a fact that will follow easily from the next proposition.

**Proposition 5.1.** Let  $A$  be a unital  $C^*$ -algebra, let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property, and let  $p, q \in A^\alpha$  be projections that are Murray-von Neumann equivalent in  $A$ . Then,  $p$  and  $q$  are Murray-von Neumann equivalent in  $A^\alpha$ .

*Proof.* Choose  $s$  in  $A$  satisfying  $s^*s = p$  and  $ss^* = q$ . Choose a positive number  $\delta > 0$ , a partition of unity  $(f_j)_{j=1}^n$  of  $\mathbb{T}$  and a unitary  $u$  in  $A$  as in the assumptions of Theorem 4.3 for the choices of tolerance  $\varepsilon = \frac{1}{6}$  and  $F = \{s, s^*, p, q\}$ . Denote by  $\sigma: A \rightarrow A^\alpha$  the linear map given by

$$\sigma(a) = E \left( \sum_{j=1}^n f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \right)$$

for all  $a$  in  $A$ . Then

$$\sigma(p) = E \left( \sum_{j=1}^n f_j(u)^{1/2} p f_j(u)^{1/2} \right) \approx_\varepsilon p,$$

and similarly with  $q$ . Moreover,

$$\sigma(s^*)\sigma(s) \approx_{5\varepsilon} \sigma(s^*s) = \sigma(p) \approx_\varepsilon p,$$

so  $\|\sigma(s^*)\sigma(s) - p\| < 6\varepsilon = 1$ . Similarly,  $\|\sigma(s)\sigma(s^*) - q\| < 1$ . Now Lemma 2.5.3 in [Lin01] implies that there exists a partial isometry  $w$  in  $A^\alpha$  such that  $w^*w = p$  and  $ww^* = q$ . This finishes the proof.  $\square$

**Theorem 5.2.** Let  $A$  be a unital  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property.

- (1) The inclusion  $\iota: A^\alpha \hookrightarrow A$  induces an injective map  $K_*(\iota): K_*(A^\alpha) \rightarrow K_*(A)$ .
- (2) Assume that  $K_j(A)$  is finitely generated for some  $j = 0, 1$ . Then there exists a group homomorphism  $\pi: K_j(A) \rightarrow K_j(A^\alpha)$  such that  $\pi \circ K_j(\iota) = \text{id}_{K_j(A^\alpha)}$ .

*Proof.* Part (1). The result for  $K_1$  follows from the result for  $K_0$  by replacing  $(A, \alpha)$  with  $(A \otimes B, \alpha \otimes \text{id}_B)$ , where  $B$  is any unital  $C^*$ -algebra satisfying the UCT such that  $K_0(B) = 0$  and  $K_1(B) = \mathbb{Z}$ , and using the Künneth formula.

Let  $x \in K_0(A^\alpha)$  such that  $K_0(\iota)(x) = 0$ . Choose  $p$  and  $q$  in  $M_\infty(A^\alpha)$  with  $x = [p] - [q]$ , so that  $[p] = [q]$  in  $K_0(A)$ . By replacing  $p$  or  $q$  by  $p \oplus 1_k$  or  $q \oplus 1_k$  respectively, we may assume that there exists  $n \in \mathbb{N}$  such that  $p$  and  $q$  belong to  $M_n(A)^\alpha$ . Also, by replacing  $A$  by  $M_n(A)$ , we can assume that  $p$  and  $q$  are Murray-von Neumann equivalent in  $M_n(A)$ . Since the induced action of  $\alpha$  on  $M_n(A)$  has the Rokhlin property, it follows from Proposition 4.1 that  $p$  and  $q$  are Murray-von Neumann equivalent in  $M_n(A)^\alpha$ , and thus  $x = 0$  in  $K_0(A^\alpha)$ .

Part (2). Assume that  $K_0(A)$  is finitely generated. Then there exist nonnegative integers  $t$  and  $n_0$ , positive integers  $n_1, \dots, n_t$ , and a group isomorphism

$$K_0(A) \cong \mathbb{Z}^{n_0} \oplus \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_t}.$$

From now on, we will simply identify  $K_0(A)$  with  $\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ . For  $j = 1, \dots, n_0$ , denote by  $1_j$  the unit of the  $j$ -th direct summand of the free part of  $K_0(A)$ , and for  $k = 1, \dots, t$ , denote by  $1'_k$  the unit of the  $k$ -th direct summand  $\mathbb{Z}_{n_k}$  of the torsion part of  $K_0(A)$ . Choose  $N$  in  $\mathbb{N}$  and projections  $p_1, \dots, p_{n_0}, q_1, \dots, q_{n_0}$  and  $p'_1, \dots, p'_t, q'_1, \dots, q'_t$  in  $M_N(A)$  such that

$$1_j = [p_j] - [q_j] \quad \text{and} \quad 1'_k = [p'_k] - [q'_k]$$

for all  $j = 1, \dots, n_0$  and all  $k = 1, \dots, t$ . For  $k = 1, \dots, t$ , choose  $N_k$  in  $\mathbb{N}$  and a partial isometry  $v_k$  in  $M_{N+N_k}(A)$  such that

$$v_k^* v_k = \begin{pmatrix} p'_k & & & \\ & \ddots & & \\ & & p'_k & \\ & & & 1_{N_k} \end{pmatrix} \quad \text{and} \quad v_k v_k^* = \begin{pmatrix} q'_k & & & \\ & \ddots & & \\ & & q'_k & \\ & & & 1_{N_k} \end{pmatrix}.$$

(There are  $n_k$  repetitions of each of  $p'_k$  and  $q'_k$ .) In other words,  $v_k$  witnesses the fact that  $[p'_k] - [q'_k]$  has order  $n_k$  in  $K_0(A)$ . Denote by  $P$  and  $V$  the subsets of  $M_N(A)$  given by

$$P = \{p_j, q_j, p'_k, q'_k : j = 1, \dots, n_0, k = 1, \dots, t\} \quad \text{and} \quad V = \{v_k, v_k^* : k = 1, \dots, t\},$$

and set  $F = P \cup V$ . Choose  $M$  in  $\mathbb{N}$  so that  $F \subseteq M_M(A)$ , and note that the induced action  $\alpha^{(M)}$  of  $\alpha$  on  $M_M(A)$  has the Rokhlin property by Proposition 3.2 in [Gar14a]. Choose a positive number  $\delta > 0$ , a partition of unity  $(f_j)_{j=1}^n$  in  $\mathbb{T}$ , and a unitary  $u$  in  $A$  as in the assumptions of Theorem 4.3 for the action  $\alpha^{(M)}$ , with tolerance  $\varepsilon = \frac{1}{15}$  and finite set  $F$ . Denote by  $\sigma : M_M(A) \rightarrow M_M(A^\alpha)$  the positive linear map given by

$$\sigma(a) = E \left( \sum_{j=1}^n f_j(u)^{1/2} \alpha_{\zeta_j}^{(M)}(a) f_j(u)^{1/2} \right)$$

for all  $a$  in  $M_M(A)$ . For  $p$  in  $P$ , we have  $\sigma(p)^* = \sigma(p)$  and

$$\|\sigma(p)^2 - \sigma(p)\| < 5\varepsilon.$$

Use Lemma 2.5.5 in [Lin01] to choose, for each  $j = 1, \dots, n_0$  and each  $k = 1, \dots, t$ , projections  $r_j, s_j, r'_k$  and  $s'_k$  in  $A^\alpha$ , such that

$$\begin{aligned} \|r_j - \sigma(p_j)\| &< 2\|\sigma(p_j)^2 - \sigma(p_j)\| < 10\varepsilon; \\ \|s_j - \sigma(q_j)\| &< 2\|\sigma(q_j)^2 - \sigma(q_j)\| < 10\varepsilon, \end{aligned}$$

and similarly with  $\|r'_k - \sigma(p'_k)\|$  and  $\|s'_k - \sigma(q'_k)\|$ .

Define a map  $\pi : K_0(A) \rightarrow K_0(A^\alpha)$  given on generators by

$$1_j \mapsto [r_j] - [s_j] \quad \text{and} \quad 1'_k \mapsto [r'_k] - [s'_k].$$

We claim that  $\pi$  is well-defined, that is, that it preserves the order of the torsion elements. We must check that for  $k$  in  $\{1, \dots, t\}$ , we have  $n_k[r'_k] = n_k[s'_k]$  in  $K_0(A^\alpha)$ . Fix  $k$  in  $\{1, \dots, t\}$ . Then

$$\sigma(v_k)^* \sigma(v_k) \approx_{5\varepsilon} \begin{pmatrix} \sigma(p'_k) & & & \\ & \ddots & & \\ & & \sigma(p'_k) & \\ & & & 1_{N_k} \end{pmatrix} \approx_{10\varepsilon} \begin{pmatrix} r'_k & & & \\ & \ddots & & \\ & & r'_k & \\ & & & 1_{N_k} \end{pmatrix},$$

so  $\|\sigma(v_k)^* \sigma(v_k) - \text{diag}(r'_k, \dots, r'_k, 1_{N_k})\| < 1$ . Similarly,

$$\left\| \sigma(v_k) \sigma(v_k)^* - \begin{pmatrix} s'_k & & & \\ & \ddots & & \\ & & s'_k & \\ & & & 1_{N_k} \end{pmatrix} \right\| < 1.$$

Lemma 2.5.3 in [Lin01] implies that there is a partial isometry in  $M_{N+N_k}(A)$  close to  $\hat{v}_k$ , that witnesses the fact that  $[r'_k] - [s'_k]$  has order  $n_k$  in  $K_0(A^\alpha)$ . This shows that the map  $\pi: K_0(A) \rightarrow K_0(A^\alpha)$  is well-defined. To check that it is a splitting for  $K_0(\iota)$ , it is enough to notice that if  $p$  is a projection in  $M_\infty(A^\alpha)$ , then  $\pi([p]) = [p]$ , since the averaging that defines  $\pi$  leaves all the elements of  $M_\infty(A^\alpha)$  fixed.

The argument for  $K_1$  is analogous and is omitted.  $\square$

Even if none of the  $K$ -groups of  $A$  is finitely generated, there is something that can be said in general about the (isomorphic) image of  $K_*(A^\alpha)$  in  $K_*(A)$ . We recall the definition of a pure subgroup and a pure extension.

**Definition 5.3.** Let  $G$  be an abelian group and let  $G'$  be a subgroup. We say that  $G'$  is *pure* if every torsion element of  $G/G'$  lifts to a torsion element of the same order in  $G$ . Equivalently,  $nG' = nG \cap G'$  for all  $n$  in  $\mathbb{N}$ .

An extension  $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$  is said to be *pure* if  $G'$  is a pure subgroup of  $G$ .

One checks that a subgroup  $G'$  of an abelian group  $G$  is pure if and only if the following holds: for every finitely generated subgroup  $H''$  of  $G/G'$ , if  $H$  denotes the preimage of  $H''$  under the canonical quotient map  $G \rightarrow G/G'$ , then the induced extension

$$0 \rightarrow G' \rightarrow H \rightarrow H'' \rightarrow 0$$

splits. The proof of part (2) of Theorem 5.2 shows that the extension

$$0 \longrightarrow K_*(A^\alpha) \xrightarrow{K_*(\iota)} K_*(A) \longrightarrow K_{*+1}(A^\alpha) \longrightarrow 0$$

is always pure. In particular, it splits whenever the  $K$ -groups of  $A$  (and hence, the  $K$ -groups of  $A^\alpha$ ) are finitely generated.

It is not true that every pure extension splits. A classical example is the short exact sequence

$$0 \rightarrow \mathbb{Z}^\infty \rightarrow \mathbb{Z}^\infty \rightarrow \mathbb{Q} \rightarrow 0$$

associated to a free resolution of  $\mathbb{Q}$ . Somewhat more surprisingly, there are examples of pure subgroups which are finitely generated, yet not a direct summand (despite being a direct summand in every finitely generated subgroup that contains it). We are thankful to Derek Holt for providing the following example.

**Example 5.4.** Let  $G_1 = \mathbb{Z}$ , which we regard as the free group on the generator  $x$ , and let  $G_2$  be the abelian group on generators  $\{x, y_n : n \in \mathbb{N}\}$  and relations

$$2y_{n+1} = y_n + x$$

for all  $n$  in  $\mathbb{N}$ . Regard  $G_1$  as a subgroup of  $G_2$  via the obvious inclusion. We claim that  $G_1$  is a pure subgroup of  $G_2$ .

For  $n$  in  $\mathbb{N}$ , denote by  $G_2^{(n)}$  the subgroup of  $G_2$  generated by  $x$  and  $y_n$ . Then  $G_2^{(n)}$  is a free abelian group of rank 2, and  $G_2^{(n)} \subseteq G_2^{(n+1)}$  for all  $n$  in  $\mathbb{N}$ . Let  $H$  be a finitely generated subgroup of  $G_2$  containing  $x$ . Then there exists  $N$  in  $\mathbb{N}$  such that  $H \subseteq G_2^{(N)} \cong \mathbb{Z}x \oplus \mathbb{Z}y_N$ . Set  $H'' = H/G_1$ . Then  $H''$  is a subgroup of  $\mathbb{Z}y_N$ , and thus it is free. In particular, the extension

$$0 \rightarrow G_1 \rightarrow H \rightarrow H'' \rightarrow 0$$

splits. This shows that  $G_1$  is a pure subgroup of  $G_2$ .

Finally, we claim that  $G_1$  is not a direct summand in  $G_2$ . Assume that it is, and let  $G$  be a direct complement of  $G_1$  in  $G_2$ . Denote by  $\pi: G_2 \rightarrow G$  the canonical quotient map, and by  $\iota: G \rightarrow G_2$  the canonical inclusion. For every  $n$  in  $\mathbb{N}$ , set

$$y'_n = (\iota \circ \pi)(y_n),$$

which is an element of  $G$ . Then  $2y'_n = y'_{n-1}$  for  $n \geq 2$ . Moreover, for every  $n$  in  $\mathbb{N}$ , there exists  $k_n$  in  $\mathbb{Z}$  such that

$$y'_n = y_n + k_n x.$$

Now, the identities

$$y'_{n-1} = 2y'_n = 2y_n + 2k_n x = y_{n-1} + x + k_n x = y_{n-1} + (2k_n + 1)x,$$

imply that  $k_{n-1} = 2k_n + 1$  for all  $n \geq 2$ . Thus  $k_1 = 2^n k_{n+1} + 1$ , and hence  $k_1 - 1$  is divisible by  $2^n$  for all  $n$  in  $\mathbb{N}$ . This is a contradiction, which shows that  $G_1$  does not have a direct complement in  $G_2$ .

The example constructed above will be relevant in Section 6 of [Gar14b], where we will show that there exist circle actions with the Rokhlin property that do not have the continuous Rokhlin property, even on Kirchberg algebras that satisfy the UCT.

**Theorem 5.5.** Let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action of the circle on a unital  $C^*$ -algebra  $A$  with the Rokhlin property. Assume that either  $K_0(A)$  or  $K_1(A)$  is finitely generated. Then both  $K_0(A)$  and  $K_1(A)$  are finitely generated and there are isomorphisms  $K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$  such that the class of the unit  $[1_A] \in K_0(A)$  is sent to  $[(1_{A^\alpha}, 0)] \in K_0(A^\alpha) \oplus K_1(A^\alpha)$ .

*Proof.* Assume that  $K_0(A)$  is finitely generated. It follows from Proposition 5.2 that  $K_0(A^\alpha)$  is a direct summand of  $K_0(A)$ . In order to show that there is an isomorphism  $K_0(A^\alpha) \oplus K_1(A^\alpha) \cong K_0(A)$ , it suffices to show that the factor  $K_0(A)/K_0(A^\alpha)$  is isomorphic to  $K_1(A^\alpha)$ .

Consider the Pimsner-Voiculescu exact sequence for  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  (see Subsection 10.6 in [Bla98]):

$$\begin{array}{ccccc}
 K_0(A \rtimes_{\alpha} \mathbb{T}) & \xrightarrow{1-K_0(\widehat{\alpha})} & K_0(A \rtimes_{\alpha} \mathbb{T}) & \longrightarrow & K_0(A) \\
 \uparrow & & & & \downarrow \\
 K_1(A) & \longleftarrow & K_1(A \rtimes_{\alpha} \mathbb{T}) & \xleftarrow{1-K_1(\widehat{\alpha})} & K_1(A \rtimes_{\alpha} \mathbb{T}).
 \end{array}$$

It follows from Corollary 3.8 that the above sequence splits into two short exact sequences

$$0 \rightarrow K_j(A \rtimes_{\alpha} \mathbb{T}) \rightarrow K_j(A) \rightarrow K_{1-j}(A \rtimes_{\alpha} \mathbb{T}) \rightarrow 0$$

for  $j = 0, 1$ . Moreover, since  $A \rtimes_{\alpha} \mathbb{T}$  is Morita equivalent to  $A^{\alpha}$  by Remark 3.9, there are isomorphisms  $K_j(A \rtimes_{\alpha} \mathbb{T}) \cong K_j(A^{\alpha})$  for  $j = 0, 1$ , under which the above inclusion  $K_j(A \rtimes_{\alpha} \mathbb{T}) \rightarrow K_j(A)$  corresponds to the map on  $K$ -theory induced by the canonical inclusion  $A^{\alpha} \hookrightarrow A$ . The map on  $K_0$  has a splitting by Proposition 5.2, and hence it follows that  $K_0(A)/K_0(A^{\alpha})$  is isomorphic to  $K_1(A^{\alpha})$ . This shows that there is an isomorphism  $K_0(A) \cong K_0(A^{\alpha}) \oplus K_1(A^{\alpha})$ . In particular, the groups  $K_0(A^{\alpha})$  and  $K_1(A^{\alpha})$  are finitely generated. The short exact sequence

$$0 \rightarrow K_1(A^{\alpha}) \rightarrow K_1(A) \rightarrow K_0(A^{\alpha}) \rightarrow 0$$

forces  $K_1(A)$  to be finitely generated as well. Another application of Proposition 5.2, together with the short exact sequence above, yields an isomorphism  $K_1(A) \cong K_1(A^{\alpha}) \oplus K_0(A^{\alpha})$ , as desired.

Finally, since the unit of  $A$  belongs to  $A^{\alpha}$ , it is clear that the isomorphism  $K_0(A) \cong K_0(A^{\alpha}) \oplus K_1(A^{\alpha})$  sends the class of the unit of  $A$  in  $K_0(A)$  to  $([1_{A^{\alpha}}], 0)$  in  $K_0(A^{\alpha}) \oplus K_1(A^{\alpha})$ .  $\square$

## 6. CLASSIFICATION OF ROKHLIN ACTIONS ON KIRCHBERG ALGEBRAS

This section contains our main results concerning the classification of circle actions with the Rokhlin property on Kirchberg algebras. We begin by showing that pure infiniteness is preserved under formation of crossed products and passage to fixed point algebras by such actions (see Theorem 6.3), and that a similar conclusion holds for Kirchberg algebras (see Theorem 6.4). Moreover, in this case the predual automorphism is necessarily aperiodic (see Proposition 6.5). Finally, Theorem 6.7 and Theorem 6.9 are our main results.

We begin by recalling some standard definitions.

**Definition 6.1.** Let  $A$  be a simple  $C^*$ -algebra.

- (1) We say that  $A$  is *purely infinite*, if for every  $a$  and  $b$  in  $A$  with  $a \neq 0$ , there are  $x$  and  $y$  in  $A$  such that  $xay = b$ .
- (2) We say that  $A$  is a *Kirchberg algebra*, if it is purely infinite, separable and nuclear.

**Remark 6.2.** Condition (1) in the definition above makes sense for arbitrary  $C^*$ -algebras, and it is easy to check that it implies simplicity. Moreover, it is immediate that if  $A$  is unital, then it is purely infinite if and only if for every  $a$  in  $A$  with  $a \neq 0$ , there are  $x$  and  $y$  in  $A$  such that  $xay = 1$ .

It is well known (see Corollary 4.6 in [JO98]) that reduced crossed products by pointwise outer actions of discrete groups of purely infinite simple  $C^*$ -algebras are again purely infinite and simple. The analogous statement for locally compact groups, or even compact groups, is, however, not true. For example, the gauge action  $\gamma$  of  $\mathbb{T}$  on the Cuntz algebra  $\mathcal{O}_\infty$ , given by  $\gamma_\zeta(s_j) = \zeta s_j$  for all  $\zeta$  in  $\mathbb{T}$  and all  $j$  in  $\mathbb{N}$ , is pointwise outer by the Theorem in [MT93], and its crossed product  $\mathcal{O}_\infty \rtimes_\gamma \mathbb{T}$  is a non-simple AF-algebra, so it is far from being (simple and) purely infinite.

Hence, even though circle actions with the Rokhlin property are easily seen to be pointwise outer, this does not by itself imply that these actions preserve pure infiniteness in the simple case. To prove the theorem below, we will check the definition of pure infiniteness directly.

**Theorem 6.3.** Let  $A$  be a unital simple purely infinite  $C^*$ -algebra, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then  $A^\alpha$  and  $A \rtimes_\alpha \mathbb{T}$  are purely infinite simple  $C^*$ -algebras.

*Proof.* Since  $A^\alpha \otimes \mathcal{K}(L^2(\mathbb{T}))$  and  $A \rtimes_\alpha \mathbb{T}$  are isomorphic by Corollary 3.12, it is enough to prove that  $A^\alpha$  is purely infinite simple.

Let  $a \in A^\alpha$  be nonzero and let  $\varepsilon = \frac{1}{100}$ . Since  $A$  is purely infinite simple, there exist  $x, y \in A$  such that  $xay = 1$ . One can assume that  $\|a\| = 1$  and hence that  $\|x\| < 1 + \varepsilon$  and  $\|y\| < 1 + \varepsilon$  by Lemma 4.1.7 in [Rør02]. To show that  $A^\alpha$  is purely infinite simple, it will be enough to find  $\hat{x}$  and  $\hat{y}$  in  $A^\alpha$  such that  $\hat{x}\hat{a}\hat{y}$  is invertible.

Choose a positive number  $\delta > 0$ , a partition of unity  $(f_j)_{j=1}^n$  in  $C(\mathbb{T})$ , and a unitary  $u$  in  $A$  as in the assumptions of Theorem 4.3 for the choice of tolerance  $\varepsilon$  and compact set  $F = \{x, y, a\}$ . Denote by  $\sigma: A \rightarrow A^\alpha$  the positive map defined in Theorem 4.3. Then

$$\sigma(a) = E \left( \sum_{j=1}^n f_j(u)^{1/2} a f_j(u)^{1/2} \right) \approx_\varepsilon a.$$

Moreover,

$$\begin{aligned} \sigma(x)a\sigma(y) &\approx_{(1+\varepsilon)^2\varepsilon} \sigma(x)\sigma(a)\sigma(y) \\ &\approx_{\varepsilon(1+1+\varepsilon+3)} \sigma(xa)\sigma(y) \\ &\approx_{\varepsilon(1+1+\varepsilon+3)} \sigma(xay) = 1. \end{aligned}$$

Hence  $\|\sigma(x)a\sigma(y) - 1\| < \varepsilon^3 + 2\varepsilon^2 + 11\varepsilon < 1$ . It follows that  $\sigma(x)a\sigma(y)$  is invertible, and hence  $A^\alpha$  is purely infinite simple.  $\square$

**Theorem 6.4.** Let  $A$  be a unital Kirchberg algebra, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then  $A^\alpha$  and  $A \rtimes_\alpha \mathbb{T}$  are Kirchberg algebras.

*Proof.* It is well-known that  $A^\alpha$  and  $A \rtimes_\alpha \mathbb{T}$  are both separable and nuclear. Simplicity of both follows from Proposition 3.10. Pure infiniteness follows from Theorem 6.3.  $\square$

If  $A$  is a  $C^*$ -algebra and  $\varphi$  is an automorphism of  $A$ , we say that  $\varphi$  is *aperiodic* if  $\varphi^n$  is not inner for all  $n$  in  $\mathbb{N}$ .

Recall that the center of a simple unital  $C^*$ -algebra is trivial.

**Proposition 6.5.** Let  $A$  be a simple unital  $C^*$ -algebra, and let  $\varphi$  be an automorphism of  $A$ . Then  $\varphi$  is aperiodic if and only if  $A \rtimes_\varphi \mathbb{Z}$  is simple.



*Proof.* If  $\varphi$  is aperiodic, it follows from Theorem 3.1 in [Kis81] that the crossed product  $A \rtimes_{\varphi} \mathbb{Z}$  is simple.

Conversely, suppose that there exist  $n$  in  $\mathbb{N}$  and a unitary  $v$  in  $A$  such that  $\varphi^n = \text{Ad}(v)$ . Set

$$w = v\varphi(v) \cdots \varphi^{n-1}(v).$$

Then  $\varphi^{n^2} = \text{Ad}(w)$ , so  $\varphi^{n^2}$  is also inner. Moreover, it follows from the fact that  $v$  is  $\varphi^n$ -invariant that  $w$  is  $\varphi$ -invariant. With  $u$  denoting the canonical unitary in  $A \rtimes_{\varphi} \mathbb{Z}$  that implements  $\varphi$ , we have  $uwu^* = w$  in  $A \rtimes_{\varphi} \mathbb{Z}$ . Set  $z = u^{n^2}w^*$ . We claim that  $z$  is a unitary in the center of  $A \rtimes_{\alpha} \mathbb{Z}$ .

It is clear that  $z$  commutes with  $u$ , and for  $a$  in  $A$  we have

$$zaz^* = u^{n^2}w^*aw(u^{n^2})^* = u^{n^2}\alpha^{-n^2}(a)(u^{n^2})^* = a,$$

so the claim follows.

Since the center of  $A \rtimes_{\alpha} \mathbb{Z}$  is trivial, there is a complex number  $\lambda$  with  $|\lambda| = 1$  such that  $u^{n^2} = \lambda w$ . In particular,  $u^{n^2}$  belongs to  $A$ , which is a contradiction. This shows that  $\varphi$  is aperiodic.  $\square$

**Definition 6.6.** Let  $A$  and  $B$  be  $C^*$ -algebras, and let  $\varphi$  and  $\psi$  be automorphisms of  $A$  and  $B$  respectively. We say that  $\varphi$  and  $\psi$  are *KK-conjugate*, if there exists an invertible element  $x$  in  $KK(A, B)$  such that  $[1_A] \times x = [1_B]$  and  $KK(\psi) \cdot x = x \cdot KK(\varphi)$ .

**Theorem 6.7.** Let  $A$  and  $B$  be unital Kirchberg algebras, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  and  $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$  be actions with the Rokhlin property. Denote by  $\check{\alpha}$  and  $\check{\beta}$  the predual automorphisms of  $\alpha$  and  $\beta$  respectively. (See Theorem 3.11.) Then  $\alpha$  and  $\beta$  are conjugate if and only if  $\check{\alpha}$  and  $\check{\beta}$  are *KK-conjugate*.

*Proof.* Assume that  $\alpha$  and  $\beta$  are conjugate, and let  $\theta: A \rightarrow B$  be an isomorphism such that  $\theta \circ \alpha_{\zeta} = \beta_{\zeta} \circ \theta$  for all  $\zeta$  in  $\mathbb{T}$ . Then  $\theta$  maps  $A^{\alpha}$  onto  $B^{\beta}$ . Denote by  $\phi: A^{\alpha} \rightarrow B^{\beta}$  the restriction of  $\theta$  to  $A^{\alpha}$ . Then  $\phi$  is an isomorphism, and  $KK(\phi)$  is invertible in  $KK(A^{\alpha}, B^{\beta})$ .

Denote by  $u$  the canonical unitary in  $A \cong A^{\alpha} \rtimes_{\check{\alpha}} \mathbb{Z}$  that implements  $\check{\alpha}$ , and likewise, denote by  $v$  the canonical unitary in  $B \cong B^{\beta} \rtimes_{\check{\beta}} \mathbb{Z}$  that implements  $\check{\beta}$ . Set  $w = v\theta(u)^*$ , which is a unitary in  $B$ . We claim that  $w$  belongs to  $B^{\beta}$ . To see this, note that if  $\zeta$  belongs to  $\mathbb{T}$ , then

$$\beta_{\zeta}(w) = \beta_{\zeta}(v\theta(u)^*) = \beta_{\zeta}(v)\theta(\alpha_{\zeta}(u))^* = \zeta v \bar{\zeta} \theta(u)^* = v\theta(u)^* = w,$$

which proves the claim.

Given  $a$  in  $A^{\alpha}$ , we have

$$\begin{aligned} (\text{Ad}(w) \circ \phi \circ \check{\alpha})(a) &= w(\theta(uau^*))w^* \\ &= w\theta(u)\phi(a)\theta(u)^*w^* \\ &= v\phi(a)v^* \\ &= (\text{Ad}(v) \circ \phi)(a) \\ &= (\check{\beta} \circ \phi)(a). \end{aligned}$$

In particular,  $\phi \circ \check{\alpha}$  and  $\check{\beta} \circ \phi$  are unitarily equivalent, and thus  $KK(\phi)$  is a *KK*-equivalence between  $A^{\alpha}$  and  $B^{\beta}$  intertwining  $KK(\check{\alpha})$  and  $KK(\check{\beta})$ . This shows the “only if” implication.

Conversely, assume that  $\check{\alpha}$  and  $\check{\beta}$  are  $KK$ -conjugate, and let  $x \in KK(A^\alpha, B^\beta)$  be an invertible element implementing the equivalence. Since  $A^\alpha$  and  $B^\beta$  are Kirchberg algebras by Theorem 6.4, it follows from Theorem 4.2.1 in [Phi00] that there exists an isomorphism  $\phi: A^\alpha \rightarrow B^\beta$  such that  $KK(\phi) = x$ . Thus,  $\phi \circ \check{\alpha} \circ \phi^{-1}$  and  $\check{\beta}$  determine the same class in  $KK(B^\beta, B^\beta)$ . Now, since  $A$  and  $B$  are simple, it follows from Proposition 6.5 that  $\check{\alpha}$  and  $\check{\beta}$  are aperiodic, and, consequently, they are cocycle conjugate by Theorem 5 in [Nak00]. In particular,  $\check{\alpha}$  and  $\check{\beta}$  are exterior conjugate. Finally, Proposition 2.9 implies that the dual actions of  $\check{\alpha}$  and  $\check{\beta}$ , which are themselves conjugate to  $\alpha$  and  $\beta$ , respectively, are conjugate. This finishes the proof.  $\square$

As a consequence, we can show that any two circle actions with the Rokhlin property on  $\mathcal{O}_2$  are conjugate. The same result was obtained with completely different methods in Corollary 8.8 in [Gar14a].

**Corollary 6.8.** Let  $\alpha$  and  $\beta$  be circle actions with the Rokhlin property on  $\mathcal{O}_2$ . Then  $\alpha$  and  $\beta$  are conjugate.

*Proof.* The fixed point algebras  $(\mathcal{O}_2)^\alpha$  and  $(\mathcal{O}_2)^\beta$  are Kirchberg algebras by Theorem 6.4, and absorb  $\mathcal{O}_2$  by Corollary 3.4 in [HW07], so  $(\mathcal{O}_2)^\alpha \cong (\mathcal{O}_2)^\beta \cong \mathcal{O}_2$  by Theorem 3.8 in [KP00]. Moreover,  $KK(\check{\alpha})$  and  $KK(\check{\beta})$  are both trivial since  $KK(\mathcal{O}_2, \mathcal{O}_2) = 0$ . It follows from Theorem 6.7 that  $\alpha$  and  $\beta$  are conjugate.  $\square$

We point out that Corollary 3.4 of [HW07], which is also used in the proof of Corollary 8.8 in [Gar14a], is not really necessary to conclude that the fixed point algebras of  $\alpha$  and  $\beta$  are isomorphic to  $\mathcal{O}_2$ . Indeed,  $(\mathcal{O}_2)^\alpha$  and  $(\mathcal{O}_2)^\beta$  have trivial  $K$ -theory by Theorem 5.2, and if we knew that they satisfy the UCT, we would conclude that they are isomorphic to  $\mathcal{O}_2$  by classification. However, it follows from Corollary 6.6 in [Gar14b] that  $\alpha$  and  $\beta$  have the continuous Rokhlin property, and then the UCT for their fixed point algebras follows from Theorem 3.12 in [Gar14b].

In the presence of the UCT, the invariant takes a more manageable form.

**Theorem 6.9.** Let  $A$  and  $B$  be unital Kirchberg algebras, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  and  $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$  be actions with the Rokhlin property. Assume that  $A^\alpha$  and  $B^\beta$  satisfy the Universal Coefficient Theorem. Denote by  $\iota_A: A^\alpha \rightarrow A$  and  $\iota_B: B^\beta \rightarrow B$  the canonical inclusions. Then  $\alpha$  and  $\beta$  are conjugate if and only if there are  $\mathbb{Z}_2$ -graded group isomorphisms

$$\varphi_*: K_*(A) \rightarrow K_*(B) \quad \text{and} \quad \psi_*: K_*(A^\alpha) \rightarrow K_*(B^\beta),$$

with  $\varphi_0([1_A]) = [1_B]$  and  $\psi_0([1_{A^\alpha}]) = [1_{B^\beta}]$ , such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_j(A^\alpha) & \xrightarrow{K_j(\iota_A)} & K_j(A) & \longrightarrow & K_{1-j}(A^\alpha) \longrightarrow 0 \\ & & \downarrow \psi_j & & \downarrow \varphi_j & & \downarrow \psi_{1-j} \\ 0 & \longrightarrow & K_j(B^\beta) & \xrightarrow{K_j(\iota_B)} & K_j(B) & \longrightarrow & K_{1-j}(B^\beta) \longrightarrow 0 \end{array}$$

is commutative for  $j = 0, 1$ .

*Proof.* We will check that the assumptions of this theorem imply the hypotheses of Theorem 6.7.

Denote by  $\check{\alpha}$  and  $\check{\beta}$  the predual automorphisms of  $\alpha$  and  $\beta$  respectively. Then  $\check{\alpha}$  is approximately representable by Proposition 3.6, so it induces the identity map on  $K$ -theory. Consider the short exact sequence

$$0 \longrightarrow \text{Ext}(K_*(A^\alpha), K_{*+1}(A^\alpha)) \xrightarrow{\varepsilon} KK(A^\alpha, A^\alpha) \xrightarrow{\tau} \text{Hom}(K_*(A^\alpha), K_*(A^\alpha)) \longrightarrow 0,$$

coming from the UCT for the pair  $(A^\alpha, A^\alpha)$ . Then  $\tau(1 - KK(\check{\alpha})) = 0$ , and thus  $1 - KK(\check{\alpha})$  is represented by a class in  $\text{Ext}(K_*(A^\alpha), K_{*+1}(A^\alpha))$ . This extension is precisely the sum of the two short exact sequences arising from the Pimsner-Voiculescu 6-term exact sequence (that one really gets two short exact sequences follows from the fact that  $K_*(\check{\alpha}) = 1$ ). An analogous statement holds for  $B^\beta$  and  $\check{\beta}$ .

Using the UCT for  $A^\alpha$  and  $B^\beta$ , choose an invertible element  $x$  in  $KK(A^\alpha, B^\beta)$  such that  $\tau(x) = \psi_*$ . The assumptions on the maps  $\varphi_0$  and  $\varphi_1$  imply that  $x$  implements a  $KK$ -equivalence between  $1 - KK(\check{\alpha})$  and  $1 - KK(\check{\beta})$ . Hence it also implements a  $KK$ -equivalence between  $KK(\check{\alpha})$  and  $KK(\check{\beta})$ , and thus the result follows from Theorem 6.7 above.  $\square$

Adopt the notation and assumptions of Theorem 6.9. If either of the  $K$ -groups of  $A$  is finitely generated, it will follow from Corollary 6.6 and Theorem 4.6 in [Gar14b] that  $A^\alpha$  satisfies the UCT if (and only if)  $A$  does. We have, nevertheless, not been able to show that the UCT for  $A$  implies the UCT for  $A$  without further assuming that either  $\alpha$  has the *continuous* Rokhlin property (see Definition 3.1 in [Gar14b]), or that the  $K$ -groups of  $A$  are finitely generated. We formally raise this as a question.

**Question 6.10.** Let  $A$  be a separable, unital  $C^*$ -algebra, and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. If  $A$  satisfies the UCT, does it follow that  $A^\alpha$  satisfies the UCT as well?

If  $\alpha$  is assumed to have the continuous Rokhlin property, then the result will be shown to be true in Theorem 4.6 in [Gar14b]. If one replaces the circle group  $\mathbb{T}$  with a finite group, then the resulting question has a positive answer in the *nuclear* case, as was shown in Corollary 3.9 in [OP12].

**6.1. An alternative approach using Bentmann-Meyer's work.** In [BM14], Bentmann and Meyer use homological algebra to classify objects in triangulated categories that have a projective resolution of length two. Starting with a certain homological invariant, their results show that two objects with a projective resolution of length two can be classified by the invariant together with a certain obstruction class in an  $\text{Ext}^2$ -group computed from the given invariant. Their methods apply to the triangulated category  $\mathcal{KK}^\mathbb{T}$  of  $C^*$ -algebras with a circle action, where morphisms are given by elements of the equivariant  $KK$ -theory, and the homological invariant is equivariant  $K$ -theory. ( $R(\mathbb{T})$ -modules have projective resolutions of length two, since the circle group has dimension one. In general, if  $G$  is a Lie group and  $T$  is any maximal torus, then the cohomological dimension of  $R(G)$  is  $\text{rank}(T) + 1$ . See the comments below Proposition 3.1 in [BM14].)

We compute the equivariant  $K$ -theory of a circle action with the Rokhlin property in the proposition below. We show that for such actions, equivariant  $K$ -theory and  $K$ -theory of the fixed point algebra are isomorphic as  $R(\mathbb{T})$ -modules (the latter carrying the trivial  $R(\mathbb{T})$ -module structure), thus placing our results (particularly Theorem 6.9) in the homological algebra context of Bentmann-Meyer's work.

**Proposition 6.11.** Let  $A$  be a unital  $C^*$ -algebra and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  be an action with the Rokhlin property. Then there is a natural  $R(\mathbb{T})$ -module isomorphism

$$K_*^\alpha(A) \cong K_*(A^\alpha),$$

where the  $R(\mathbb{T})$ -module structure on  $K_*(A^\alpha)$  is the trivial one.

*Proof.* Recall that  $R(\mathbb{T}) \cong \mathbb{Z}[x, x^{-1}]$ . By Julg's Theorem (here reproduced as Theorem 2.7), there is a natural isomorphism  $K_*^\alpha(A) \cong K_*(A \rtimes_\alpha \mathbb{T})$ , where the  $\mathbb{Z}[x, x^{-1}]$ -module structure on  $K_*(A \rtimes_\alpha \mathbb{T})$  is determined by the dual action  $\hat{\alpha}$ , meaning that the action of  $x$  agrees with the action of  $K_*(\hat{\alpha})$ . Now,  $\hat{\alpha}$  is approximately inner by Proposition 3.8, so it induces the trivial automorphism of the  $K$ -theory. This shows that the  $R(\mathbb{T})$ -module structure on  $K_*^\alpha(A)$  is the trivial one.

Finally, there is a natural isomorphism  $K_*(A \rtimes_\alpha \mathbb{T}) \cong K_*(A^\alpha)$  by Corollary 3.12.  $\square$

Let  $A$  and  $B$  be unital  $C^*$ -algebras (not necessarily Kirchberg algebras), and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  and  $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$  be circle actions with the Rokhlin property. Assume that  $\alpha$  and  $\beta$  belong to the equivariant bootstrap class. Bentmann and Meyer show (see Subsection 3.2 in [BM14]) that in this context, the actions  $\alpha$  and  $\beta$  are  $KK^\mathbb{T}$ -equivalent if and only if there is an isomorphism  $K_*^\alpha(A) \cong K_*^\beta(B)$  that respects the elements in  $\text{Ext}_\mathbb{Z}(K_*^\alpha(A), K_{*+1}^\alpha(A))$  and  $\text{Ext}_\mathbb{Z}(K_*^\beta(B), K_{*+1}^\beta(B))$  determined by  $\alpha$  and  $\beta$  respectively.

It is shown in Proposition 3.1 in [BM14] that a circle action  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  on a unital  $C^*$ -algebra  $A$  belongs to the equivariant bootstrap class if and only if  $A$  and  $A \rtimes_\alpha \mathbb{T}$  satisfy the UCT. When  $\alpha$  has the Rokhlin property, this is equivalent to  $A^\alpha$  satisfying the UCT. Thus, the UCT assumptions in Theorem 6.9 amount to requiring the actions  $\alpha$  and  $\beta$  there to be in the equivariant bootstrap class.

We have not been able to identify the element in

$$\text{Ext}_\mathbb{Z}(K_*^\alpha(A), K_{*+1}^\alpha(A)) \cong \text{Ext}_\mathbb{Z}(K_*(A^\alpha), K_{*+1}(A^\alpha))$$

determined by  $\alpha$ . However, we suspect that under the natural identifications, and up to a sign, it must agree with the Ext class of its predual automorphism  $\check{\alpha}$ . If this were true, we would have proved the following.

**Conjecture 6.12.** Let  $A$  and  $B$  be separable, unital  $C^*$ -algebras and let  $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$  and  $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$  be circle actions with the Rokhlin property. Assume that  $A^\alpha$  and  $B^\beta$  satisfy the UCT. Then the following statements are equivalent:

- (1) The actions  $\alpha$  and  $\beta$  are  $KK^\mathbb{T}$ -equivalent;
- (2) The automorphisms  $\check{\alpha}$  and  $\check{\beta}$  are  $KK$ -conjugate;
- (3) There are group isomorphisms

$$\varphi_*: K_*(A) \rightarrow K_*(B) \quad \text{and} \quad \psi_*: K_*(A^\alpha) \rightarrow K_*(B^\beta),$$

with  $\varphi_0([1_A]) = [1_B]$  and  $\psi_0([1_{A^\alpha}]) = [1_{B^\beta}]$ , such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_j(A^\alpha) & \xrightarrow{K_j(\iota_A)} & K_j(A) & \longrightarrow & K_{1-j}(A^\alpha) \longrightarrow 0 \\ & & \downarrow \psi_j & & \downarrow \varphi_j & & \downarrow \psi_{1-j} \\ 0 & \longrightarrow & K_j(B^\beta) & \xrightarrow{K_j(\iota_B)} & K_j(B) & \longrightarrow & K_{1-j}(B^\beta) \longrightarrow 0 \end{array}$$

is commutative for  $j = 0, 1$ .

The expected result would have the advantage of holding for arbitrary separable, unital  $C^*$ -algebras  $A$  and  $B$  satisfying the UCT (not necessarily purely infinite, or even simple). In order to recover Theorem 6.9 from it, it would be enough to show that under the assumptions of Conjecture 6.12, if moreover  $A$  and  $B$  are Kirchberg algebras, then  $\alpha$  and  $\beta$  are conjugate if and only if they are  $KK^{\mathbb{T}}$ -equivalent via a  $KK^{\mathbb{T}}$ -equivalence that respects the classes of the units. Corollary 4.2.2 in [Phi00] (see also [Kir94]) suggests that one may be able to prove this directly, and maybe without even assuming that the actions have the Rokhlin property. We have, however, not explored this direction any further.

## 7. CONCLUDING REMARKS

In this final section, we motivate some connections with the second part of this work [Gar14b]. We also give some indications of the difficulties of extending the results in this paper to actions of other compact Lie groups.

**7.1. Range of the invariant and related questions.** We have shown in Theorem 6.7 that circle actions with the Rokhlin property on Kirchberg algebras are completely determined, up to conjugacy, by the pair  $(A^\alpha, KK(\bar{\alpha}))$ , consisting of the fixed point algebra  $A^\alpha$  together with the  $KK$ -class  $KK(\bar{\alpha})$  of the predual automorphism. However, we have not said anything about what pairs  $(B, x)$ , consisting of a Kirchberg algebra  $B$  and an invertible element  $x$  in  $KK(B, B)$ , arise from circle actions with the Rokhlin property as described above. There are no obvious restrictions on the  $C^*$ -algebra  $B$ , while Proposition 3.6 shows that  $x$  must belong to the kernel of the natural map  $KK(B, B) \rightarrow KL(B, B)$ . We do not know whether all such pairs are realized by a circle action with the Rokhlin property.

The second part of our work [Gar14b] addresses this question, and provides a complete answer under the additional assumption that the action have the continuous Rokhlin property. In Theorem 3.29 in [Gar14b], we show that every pair  $(B, x)$  as above arises from a circle action with the continuous Rokhlin property if and only if  $x = 1$ . In other words, the fixed point algebra is arbitrary and the predual automorphism is an arbitrary  $KK$ -trivial aperiodic automorphism.

We also show in Proposition 3.33 in [Gar14b], that all circle actions with the continuous Rokhlin property on Kirchberg algebras are “tensorially generated” by a specific one (which necessarily has the continuous Rokhlin property).

Finally, in the second part of this work we also provide a partial answer to Question 6.10, answering it affirmatively whenever  $\alpha$  has the continuous Rokhlin property. See Theorem 3.12 in [Gar14b]. Since the Rokhlin property and the continuous Rokhlin property agree on Kirchberg algebras with finitely generated  $K$ -theory by Corollary 6.6 in [Gar14b], these results can be applied to Rokhlin actions in many cases of interest.

**7.2. Beyond circle actions.** We close this article by explaining what difficulties one may encounter when trying to generalize the methods exhibited here to more general compact Lie group actions.

Bentmann-Meyer’s techniques depend heavily on the fact that  $R(\mathbb{T})$ -modules have projective resolutions of length two, essentially because the circle has dimension one. While this is also true for  $SU(2)$ , it fails for other natural examples of compact Lie groups like the two-torus  $\mathbb{T}^2$ , so their methods break down already in this case.

Our methods are no less dependent on low-dimensionality of the circle, though the dependence is slightly more subtle. For example, already in dimension two,  $(C(\mathbb{T}^2), \mathbf{Lt})$  is not equivariantly semiprojective (because  $C(\mathbb{T}^2)$  is not semiprojective), so Proposition 3.3, and hence Theorem 3.11, will not be true in general. There is another instance where one-dimensionality of the circle (or rather, the fact that its dual group  $\mathbb{Z}$  has rank one) was used, namely in the proof of Proposition 6.5. In fact, the corresponding statement for arbitrary discrete abelian groups is not true: Example 4.2.3 of [Phi87] shows that there exists an action  $\varphi$  of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on  $M_2$ , with  $M_2 \rtimes_{\varphi} (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong M_4$ , and such that  $\varphi|_{\mathbb{Z}_2 \times \{1\}}$  is inner. (For an example on a UCT Kirchberg algebra, simply tensor with  $\mathcal{O}_{\infty}$  with the trivial action.) Moreover, the classification of not necessarily pointwise outer actions of discrete groups (or even finite cyclic groups!) on Kirchberg algebras is probably a very challenging task.

The conclusion seems to be that neither approach is likely to work for general compact Lie groups, and that an eventual classification would require a rather different approach and machinery.

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